INFORMATION AND PUBLIC SECTOR DECISIONS

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Summary

The theoretical models in this thesis address questions relating to the interaction between information and decisions. The main issues are as follows: i) decisions are based on uncertain parameters, ii) parameter estimates are used for specific policy decisions, iii) policy decisions take the form of sequential reforms whose magnitude and frequency must be determined, iv) there are dynamic interactions between the properties of estimators and the performance of decision rules.

The method of investigation is by formulation of algebraic models whose properties are examined by analytic and numerical techniques.

The contribution to the knowledge of the subject is as follows: i) a well-known linear control model is extended to incorporate sequential reforms, ii) the properties of a limited class of optimal active learning strategies are described, iii) in Monte Carlo simulations, least squares estimates are not found to have desirable statistical properties when used in conjunction with active learning decision rules, iv) a number of well-known optimal tax models are extended to incorporate parameter uncertainty.

Contents

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Notation

The notation is defined where it first appears in each Chapter. The following definitions are maintained throughout.

c	cost, or cost function
k	reform cost
ℓ	labour supply
m	lump-sum transfer
p	consumer price
q	demand
r	rate of interest
t	tax
w	wage
x	target
y	instrument
z	fixed instrument
J	discounted flow of benefits
M	date of reform
R	government revenue
T	time endowment
U	evaluation function
V	indirect utility function
W	welfare function
α, β, γ	parameters
δ	discount factor
ε	additive error
ρ	rate of time preference
μ	risk
σ^2	variance of ε

Abbreviations

AL	Active learning
CE	Certainty equivalent decision rule
FOC	First order condition
OLS	Ordinary least squares
PWR	Pressurise water reactor
QAL	Quadratic active learning decision rule
RA	Risk averse decision rule
VM	Variance minimising decision rule

FOREWORD

Introduction

The models of public sector decision making which are formulated in this thesis may in general be expressed in the following way: given rationality, and given also a particular situation S, the best choice of action is A. Rationality, in the sense that "agents always act in a manner appropriate to the situation in which they find themselves" (Popper, 1967, p.361) is assumed throughout, and therefore the analysis proceeds by indicating the connection between the situation and the action. The novelty of the thesis results firstly from putting the public sector decision-maker in situations where particular emphasis is placed on information; and secondly, by postulating rather unusual types of actions, particularly related to learning, and establishing the kinds of situations for which this behaviour is appropriate.

Simon (1960) may be regarded as one of the leading exponents of models of perfectly rational decision-making. The process takes place in four stages: intelligence (or data) gathering, identification of options, assessment of consequences, and evaluation of consequences. In theory it would be best to execute these stages simultaneously so that all possible interactions between them could be captured. For example, examination of the means of evaluating consequences may indicate that some options are inferior and need not be considered: this in turn may allow the relevant data set to be more accurately identified, saving effort at the information gathering stage. However, practical considerations often dictate that the decision-making process must be broken down into tasks which may be performed independently by different institutions at different times. The considerations of information and learning which are discussed in this thesis indicate connections both between these tasks, and within tasks over time, and have implications for the desirable break-down of the decision, and subsequent co-ordination between tasks.

In economics, policy issues are often divided into tasks of parameter estimation, and policy optimisation. The analysis of economic data to produce parameter estimates frequently takes place independently of the uses to which they will be put. Hendry and Richard (1982, 1983) have set out a methodology for model selection. Their criteria for choosing between competing models include valid marginalisation and conditioning, white noise errors, theory consistency, and parsimony, but notable by its absence is the performance of decision rules based on the estimates. Similarly, Mizon and Richard (1986) use the encompassing principle to draw together the literature on nested and non-nested hypothesis testing, but the comparisons between models are on the basis of the statistical properties of their estimates, not the performance of the estimates in a given policy optimisation problem. Similarly, economic analysis of particular policy issues, such as Ahmad and Stern (1984) on Indian indirect tax reforms, make use of parameter estimates without taking account of the econometric model which produced them.

Inter-temporal connections between decisions are of particular interest in the context of the tax reform literature. Given a piecemeal approach to decision making, current decisions affect the options which will be available in the future. Henry (1974a) and (1974b), and Arrow and Fischer (1974) develop the concept of option values for irreversible decisions, but it is also applicable to decisions which may be reversed at some cost. Reform options are often limited by the availability of accurate information. Then an important criterion in evaluating the benefits of a given reform is the amount which may be learnt from it, and hence the potential which it creates for improving subsequent reforms. Popper (1967) makes this connection between information and reform quite explicitly.

> "... the piecemeal engineer knows, like Socrates, how little he knows. He knows that we can learn only from our mistakes. Accordingly, he will make his way, step by step, carefully comparing the results expected with the results achieved, and always on the look-out for the unavoidable unwanted consequences of any reform; and he will avoid undertaking any reforms of a complexity and scope which make it impossible for him to disentangle causes and effects, and to know what he is really doing".
>
> (Popper, 1967, pp.309)

Other authors have gone further in advocating particular decision strategies: Lindblom (1959) proposes "muddling through", and Braybrooke and Lindblom (1963) "disjoint incrementalism", and a recent review is provided by Hogwood and Gunn (1986). However, the objective of this thesis is not to propose some new decision strategy, but rather to indicate when co-ordination between and within decisions is required.

The Issues

The main issues are as follows: firstly, in a static context

i) Decisions are based on uncertain parameters

ii) Parameter estimates are used as inputs for specific policy decisions.

Theil (1957), and Simon (1956) provide certain equivalent theorems which establish the conditions under which uncertain parameters may be used as though they were perfectly accurate. These conditions do not apply to most standard models of public sector decisions, such as the optimal tax models described by Atkinson and Stiglitz (1980). Therefore uncertainty generally does matter, and the immediate questions are those of its significance, and the direction of effect. Instead of adjusting the policy decision for uncertainty, the same effect may be achieved by adjusting the estimating technique. Policy makers may then regard these "certainty equivalent estimates" as though they were known with certainty. A closely related question is whether, given uncertainty, a particular parameter estimate should be regarded as an upper or lower bound for a particular problem.

Secondly, in a dynamic setting

iii) The size and frequency of reforms must be determined

iv) There may be important interactions between the properties of estimators, and the performance of decision rules.

When the economic environment is changing, the frequency of policy reforms may be determined in order to keep up to date, or may be contingent upon observed changes in economic variables. When the size of reforms affects the accuracy of the information available in the future, the policy-maker behaves like Rothschild's (1974) gambler on the two-armed bandit, simultaneously playing for current returns and accumulating information on which to base future decisions. Connections between estimation and decisions in a dynamic setting may be particularly important because current decisions form part of the data set for subsequent estimates. It is quite possible to envisage combinations of decisions and estimates which get stuck at a sub-optimal position: the estimates are poor so the decision is cautious, as a result the data has little variation and is uninformative so the estimates do not improve, therefore another cautious decision is made ... and so on. The main questions arising here relate to the relative performance of different kinds of decision rules, given that they are based on particular parameter estimates; and the statistical properties of the estimates given that a particular decision rule has been applied.

The issues are explored by means of simple models, and numerical simulations. In common with many economic models, these are not intended to be realistic: they are ideal types in the sense of Weber (1904), and are designed to draw attention to specific issues of information and learning. Such simplifications are justified by Machlup (1978) "... it is better to be approximately right than absolutely lost in the labryinth of an excessively complex 'realistic' ideal-typical world" (p.220). Of course, the inferences which can be drawn from this methodology are limited. A description is provided, in general terms, of the kinds of situations where information and learning form important connections between parameter estimation and decision-making. It is neutral on questions of whether or not these circumstances actually exist. Therefore, it is only a first tentative step towards providing maxims for policy-makers, or establishing a mechanism for generating specific policy prescriptions.

Much of the discussion is in terms of abstract states and instruments. However, the underlying motivation relates to public sector decision-making, and in particular the choice of tax rates. The public sector has become one of the main participants in economic activity in both developed and less developed countries. In 1985 the UK central government expenditure made up 39% of GDP, and this proportion has grown since 1965 at an annual average rate of 2.25%. This growth in the public sector is a world-wide phenomenon, discussed in the Journal of Public Economics (no.3, 1985). The size of the public sector means that decisions affecting it are of considerable importance for the economy as a whole, and their significance is reinforced by subsequent effects in other sectors of the economy.

Outline of the Thesis

The outline of the thesis is summarised in Figure 1. The contents of the boxes are related to Simon's (1960) description of the perfectly rational decision-maker. The first two boxes, representing the types of information available, and the method used to learn from this information, correspond to Simon's intelligence gathering. Options are identified by reference to the box of constraints, and the benefits of all possible actions are assessed according to the functions in the evaluation box. Simon's concept of "assessment of consequences" does not appear explicitly but may be regarded as a preliminary step in the process of evaluation. The final decision box indicates the kinds of decisions which are appropriate in the given situation. The models in each of the chapters are represented by paths through the flow chart.

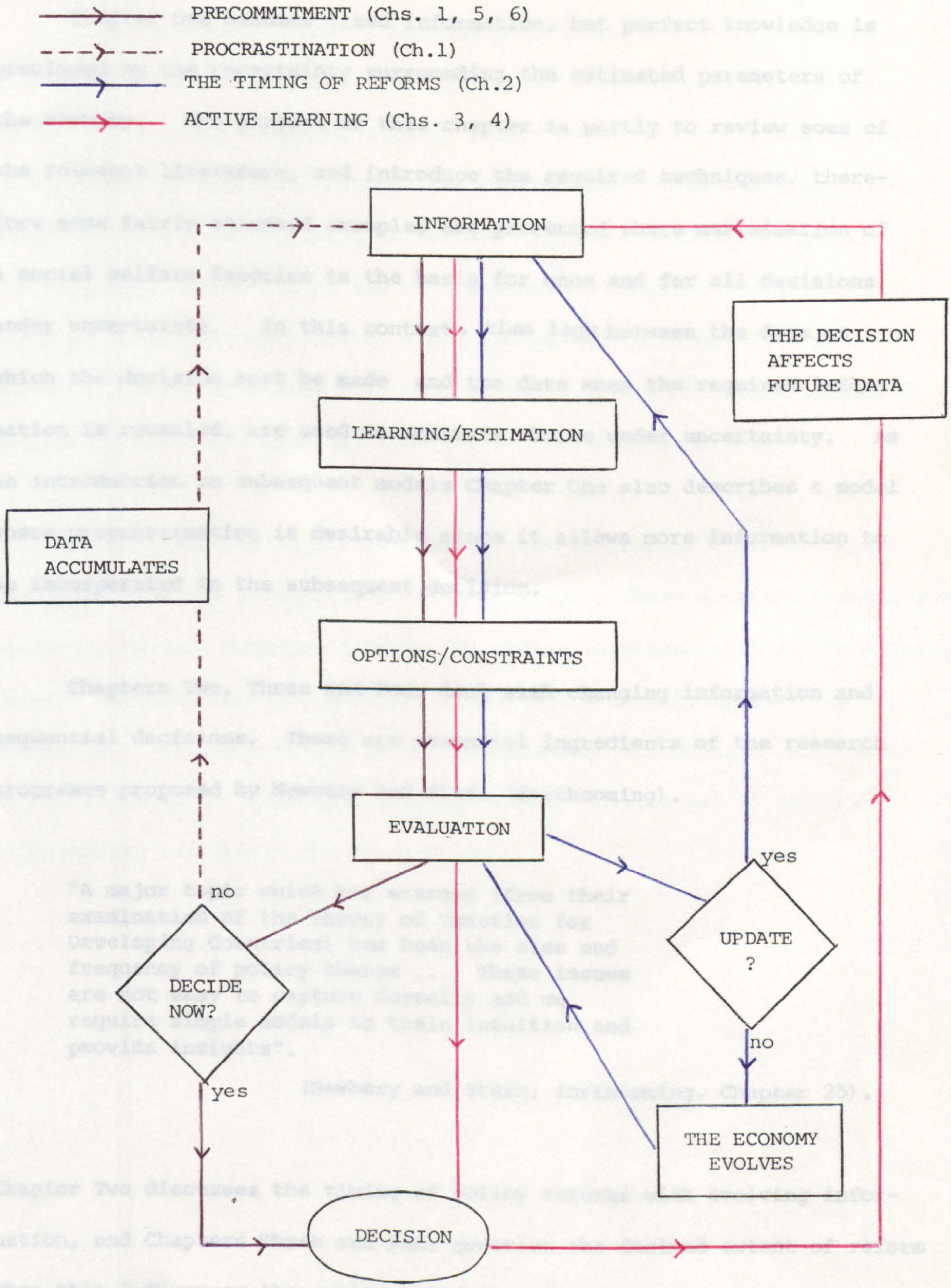

FIGURE 1 : Outline of the Thesis

Chapter One assumes fixed information, but perfect knowledge is precluded by the uncertainty surrounding the estimated parameters of the economy. The purpose of this chapter is partly to review some of the relevant literature, and introduce the required techniques, therefore some fairly standard examples are presented where maximisation of a social welfare function is the basis for once and for all decisions under uncertainty. In this context, time lags between the date on which the decision must be made and the date when the required information is revealed, are used to motivate choice under uncertainty. As an introduction to subsequent models Chapter One also describes a model where procrastination is desirable since it allows more information to be incorporated in the subsequent decision.

Chapters Two, Three and Four deal with changing information and sequential decisions. These are essential ingredients of the research programme proposed by Newbery and Stern (forthcoming).

> "A major topic which has emerged (from their examination of the Theory of Taxation for Developing Countries) has been the size and frequency of policy change ... These issues are not easy to capture formally and we require simple models to train intuition and provide insights".
>
> (Newbery and Stern, forthcoming, Chapter 25).

Chapter Two discusses the timing of policy reforms with evolving information, and Chapters Three and Four question the desired extent of reform when this influences the quality of future data.

These different types of information changes are connected with different means of learning. In Chapter Two, the policy-maker keeps up to date with the evolving economy simply by observing a state variable. On the basis of this information, the policy-maker revises his decisions according to a rule depending on whether costs are incurred by making new decisions, or in making new observations. Chapters Three and Four describe an estimation process, where the policy-maker predicts the impact which his decisions will have on the variance of future parameter estimates, and hence upon the accuracy of future information.

Chapter Six (using the methods set out in Chapter Five) returns to the model of fixed information. Whereas Chapter Three examines a situation where the optimal decision affects the information available in the future, Chapter Six asks how the quality of information affects optimal tax decisions. Existing models of optimal taxation are adjusted for uncertainty in order to judge the accuracy of the 'ideal type' of perfect information, and how it may be modified.

CHAPTER ONE

PRECOMMITMENT AND PROCRASTINATION

CHAPTER 1

SUMMARY

Precommitment is an important characteristic of a large proportion of public sector decisions. For some it is a technologically determined fact; and for others it is both a desirable and feasible property of the optimal policy.

Its significance lies in the requirement that decisions must be made before all relevant information has come to light. Two examples are presented to illustrate the possible consequences of such uncertainties. In the first it is shown that if nuclear energy is a "necessity", and uncertainty takes a particular form, greater certainty about the capabilities of nuclear technology will reduce the number of nuclear power stations which are needed. The second shows how the intertemporal allocation of a fixed budget is determined by the relative magnitudes of the rate of time preference, the expected rate of return on investments, and the degree of risk and risk aversion.

Whereas precommitment suggests that the quality of information should affect decisions, procrastination raises questions of how decisions may be used to affect the quality of information. A simple model is presented where delayed implementation allows better decisions in the future because data observations accumulate over the period of procrastination.

1.1 INTRODUCTION

Though my personal experience of the races is confined to accidental glimpses whilst changing channels on the TV on Saturday afternoons, I have it on good authority that bookmakers are completely homogeneous but punters may be divided into various categories. The bookmakers all share the characteristic that once a bet has been placed, the only way to get the money back is for the horse to win - it is no good going back ten minutes before the race having changed your mind. The punters however, are quite diverse: some place their bets well in advance and use the day of the meeting as an opportunity to socialise. Others watch the horses in the paddock, and watch the changing odds as others arrive at their decisions, and choose the best moment to put their money down. Another category of more dedicated gamblers may make any number of bets: after the start of the race they receive signals from their associates in the grandstand about which horses are running well, and which have fallen, and this information is used to revise a portfolio of bets as the race progresses.

These three kinds of behaviour are relevant to economic decision-making in an uncertain environment. The first, where the decision is made once and for all, before the state of the world is fully revealed, is called <u>precommitment</u>. The second involves choosing the right moment to make the decision. Given existing information, and given also that the date of implementation of the policy change must be announced today, at what point is the marginal cost of delay matched by the marginal benefit of improved information? The solution to this problem is the optimal degree of <u>procrastination</u>. The third, <u>irreversibility</u>, envisages incremental revisions or policy reforms, with the added restriction that changes in at least one of the instruments are irreversible.

The discussion of precommitment in this chapter is fairly standard, and serves to outline some of the techniques which are required for the model of procrastination (and for subsequent chapters), and to review some of the relevant literature. The main question is how optimal decisions are affected by the constraint that they must be made before the state of the economy is revealed. Section 1.2 discusses the literature and the techniques, and section 1.3 applies them to models determining the optimal division of a fixed public sector budget between different categories of expenditure. Section 1.4 allows the policy-maker another instrument - procrastination - and asks what implementation date the policy-maker would announce under the assumption that optimal use will be made of information which has accumulated up to that date. Concluding remarks are in section 1.5.

Issues of irreversibility have received some attention in the literature. Henry (1974a, 1974b) describes a situation where the use of an irreversible policy today closes options which would otherwise have been open tomorrow. Therefore optimal use of such instruments is more cautious the riskier is the environment. Freixas and Laffont (1984), in a more general setting, spell out precisely the assumptions required to obtain this "irreversibility effect". Precommitment and procrastination both have elements of irreversibility since they entail once-and-for-all decisions. However they exclude the vital characteristics that the policy-maker may like to revise his decisions in the light of future information. I shall return to issues of information and incremental policy reform in Chapter Two.

1.2 PRELIMINARIES

In general the objective function U may be defined on the target variable x. There may be some overlap between target variables and instruments y, but often it is convenient to define U exclusively in terms of x. This broad class of functions may be approximated with a Taylor series expansion around some arbitrary $\tilde{x}$ 1/

$$U(x) \simeq U + U_x(x-\tilde{x}) + \frac{1}{2}U_{xx}(x-\tilde{x})^2 + \frac{1}{6}U_{xxx}(x-\tilde{x})^3 + \ldots \qquad (1.1)$$

It would be simple to generalise this expression to allow a vector of target variables: such a case is considered in example one. However, some of the economic interpretations of the derivatives are meaningful only in the one dimensional case. Therefore, for the time being attention is restricted to a single target variable.

If the last two derivatives in (1.1) are insignificant then the function is locally risk neutral. It is well known in the economic texts that with linear utility functions the utility of the expected outcome is the same as the expectation of the sum of the utilities of the outcomes weighted by their probabilities of occurance. If the second derivative is negative the function is said to display risk aversion, and a decision-maker with this kind of objective would not accept fair bets, the expected outcome always being preferred to the risky alternative

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1/ For brevity, the arguments of the functions on the right-hand side have been suppressed.

with the same expected payoff.

The third derivative relates to the symmetry of objectives, and also has implications for changes in absolute and relative risk aversion. If it is zero then the objective function is locally symmetric and a small shift from the optimum in either direction brings about approximately the same decrement to utility. However, if U_{xxx} is negative (positive) then the marginal disutility of a small increase (decrease) in x above the optimum exceeds the marginal disutility of an equal reduction in x. In addition, it is straightforward to show (see Appendix 1.I) that if U_{xxx} is zero or negative, both absolute and relative risk aversion are increasing in x. Finally, a policy choice is said to be certainty equivalent if the same choice would be made under complete certainty. This term may also be applied to target variables: the certain equivalent value of the target variable is the expected outcome of a certainty equivalent policy choice. These concepts are all related to the properties of the objective function. Rational choice between alternative policies also requires well-defined beliefs about the likely consequences of any decision.

A frequently used method for analysing the consequences of actions is to allow an unknown state variable to determine what outcome will be associated with any given choice. All that is required is then a means of describing the policy-maker's beliefs about the probability that any state will arise. A convenient way to do this is to use a distribution function, the main drawback being that for tractability, the state variable must be a continuous random variable. In practice there may be situations where the state variable is discrete - the disaster either happens or it does not - and for these cases the state space approach would be more suitable. However, the models here are based exclusively on distribution functions.

The choice of distribution function raises two important issues. The first is one of dimensionality. It is assumed throughout this chapter that only one parameter is uncertain. This is a convenient assumption because existing theories, such as Diamond and Stiglitz (1974) which operate in terms of uni-dimensional risk, are directly applicable. In a more general multi-dimensional situation, it would be difficult to say when increases in some kinds of uncertainty, and reductions in others, added up to an overall increase or reduction in risk. (Criteria for making such judgements are discussed by Atkinson and Bourgignon, 1984). Having restricted attention to a single univariate distribution, the second issue is the choice of a particular function. Normal and uniform distributions are particularly convenient because they may be completely described in terms of means and variances. Then, increases in risk may be regarded as synonymous with increases in variance. For more general classes of distributions, this correspondence does not necessarily hold: an increase in variance could be dominated by changes in higher moments of the distribution and result in an overall reduction of risk. A similar point is made by Atkinson (1970): inequality indices such as the Gini coefficient only give an unambiguous ranking of distributions whose Lorenz curves do not intersect. More generally when Lorenz curves cross, an inequality index may give a lower value for a distribution which according to social welfare criteria is more equal. When risk rather than inequality is the required measure of spread of the distribution, it may be the case that a distribution function with a high variance is less risky than one with a low variance. This possibility is ruled out by distributions for which mean and variance are sufficient statistics.

Clearly, the riskiness of the distribution of the uncertain variable may have implications for the optimal policy. Its influence will depend not only on the specification of the objective function, but also on the source of uncertainty. Waud (1976) draws a distinction between additive and multiplicative uncertainty. The former is the element of risk which must be faced regardless of what decisions are made. The latter may be interpreted as parameter uncertainty, where the influence of the policy decision is imperfectly known.

Additive uncertainty may be represented

$$x = y + \varepsilon \tag{1.2}$$

where ε is distribution normally with expectation zero and variance σ^2. Then necessary conditions for policy optimisation may be derived by substituting into (1.1), having defined $\tilde{x}$ as the mean of x, taking expectations and differentiating with respect to y

$$\frac{\partial EU}{\partial y} = U_y + \frac{1}{2} U_{yyy} \sigma^2 = 0 \tag{1.3}$$

Clearly, with additive uncertainty, symmetry of the objective function would mean that the optimal decision is independent of σ^2, or in algebraic terms, $U_{yyy} = 0$ (and all higher order derivatives equal to zero) implies that y should be chosen such that $U_y = 0$. Thus Theil (1957), and Simon (1956) find that certainty equivalence follows from additive uncertainty with quadratic objectives, because these functions are symmetric. Clearly the direction in which the optimal policy would diverge from certainty equivalence depends on the sign of U_{yyy}. If it is positive (negative) the decision maker will prefer to aim high (low) since a small overshoot is preferred (inferior) to a small undershoot. Malinvaud (1969) interprets $U_{yyy} > 0$ as describing a good which is a necessity - its marginal utility decreases at a rapidly decreasing rate - and this may lead to increases in the consumption of a good as it becomes riskier. Hahn (1970) for example, discusses this proposition in the context of savings. For a public investment interpretation see Example 1.

Multiplicative uncertainty 1/ may be represented

$$x = y(1 + \eta) \tag{1.4}$$

where η has expectation zero and variance s^2. In this case, substitution into (1.1), taking expectations and differentiating yields the optimal decision rule

$$\frac{\partial EU}{\partial y} = U_y + U_{yy}ys^2 + \frac{1}{2}U_{yyy}y^2s^2 = 0 \tag{1.5}$$

1/ The distinction between additive and multiplicative uncertainty is clearly related to the choice of units. For example, if logs were taken of (1.4) it would become a case of additive uncertainty.

If the objective function is symmetric the last term is zero but even so certainty equivalence does not apply. Inspection of (1.4) reveals that the variance of x depends on the choice of y, and when the decision-maker is risk averse this dependence will induce him to aim low, i.e. choose a value for the instrument below the certainty equivalent level. Hence multiplicative uncertainty with risk aversion means that smaller y's tend to be preferred since they are associated with smaller variance in the outcome. This is the algebraic argument underlying Waud (1976)'s diagramatic discussion of additive and multiplicative uncertainty. For a simple model of intertemporal public investment decisions with uncertain rates of return, see Example 2.

1.3 PRECOMMITMENT

As a rule, larger expenditure decisions are associated with a greater degree of precommitme . The consumer can decide virtually on the spur of the moment whether or not to buy an ice cream; and can have available virtually all relevant meteorological information covering the period of consumption. However, house purchases may require commitment six months in advance, and such decisions have to be made without full knowledge of such things as geographical location of future employment. For a significant proportion of public sector decisions precommitment is simply a technological fact: the lead times on the construction of power stations, for example, may be ten years.

Precommitment is not only a characteristic of production technology, but in some cases may be a desirable policy objective in itself. The private sector faces many uncertainties which may have harmful or distorting effects on their behaviour. By announcing in advance that a particular policy will be implemented, the government may attempt to compensate for such uncertainties. In effect the government agrees to share some of the risk with the private sector, and in this sense acts as an insurance agency. Since the oil crises of the 1970's, industrialists in the UK have been aware of the unpredictable nature of the world energy markets. It could be argued that the expansion of the domestic nuclear energy programme is a means of hedging against the uncertainties of global energy-price shocks. The government, on behalf of all sectors of the economy, agrees to share this risk with the industrialists. Should another oil crisis occur, the electricity industry would be in a better position to provide a substitute for oil, and alleviate the industrial consequences. However, if oil markets remain stable, the UK economy as a whole bears the burden of excess nuclear energy capacity.

For some policy decisions precommitment plays a crucial role. It is frequently argued that for certain classes of policy choices, the policy can only be successful if the government can precommit. That is, the government must be able to put itself in a position where it is difficult to change its mind. Backus and Driffill (1985) discuss the example where inflation is the result of the private sector's inflationary expectations. If the private sector believes that the government is prepared to take the unpleasant but effective anti-inflationary medicine, then inflation will be rapidly reduced. However, if there is some chance that policy-makers will be discouraged by the unpleasant side-effects, and will only half-heartedly implement the policy, then the private sector will have little faith and inflation will persist. Indeed, if the private sector knows that the government's optimal plan is time inconsistent - that they would like to announce harsh policies but then implement soft ones - the scope for effective government intervention seems severely restricted.

The consequence, given that precommitment is either necessary, or desirable and feasible, is that decisions must be made before all relevant information has become available. Rather than attempt a review of the literature on choice under uncertainty, two specific examples are examined. They are intended to illustrate the importance of the type of uncertainty and the nature of objectives, and to provide a basis for subsequent models incorporating changes in information.

1.3.1 EXAMPLE 1

Choice of expenditure on risky public investments, with additive uncertainty and asymmetric objectives.

The government has to decide how to allocate a fixed budget R between expenditure on conventional and nuclear power stations, y_1 and y_2. The two kinds of electricity x_1 and x_2 which they produce have different characteristics (in terms of the ability to respond to transient changes in demand) so the objective function depends on both, $U(x_1,x_2)$. Conventional technology is certain but nuclear is subject to additive risk: attention is restricted to those uncertainties which are independent of the scale of investment

$$x_1 = y_1$$

$$x_2 = y_2 + \varepsilon \tag{1.6}$$

where ε is normally distributed with mean zero and variance σ^2. Using the budget constraint $R = y_1 + y_2$, it is possible to write the government's objective function in terms of the choice variable y_1 and the unknown state ε.

$$U(x_1,x_2) = U(y_1, R - y_1 + \varepsilon) \equiv w(y_1,\varepsilon) \tag{1.7}$$

Taking a Taylor expansion of w around the expectation of ε gives

$$w = w(y_1,0) + \frac{1}{2} w_{\varepsilon\varepsilon} \sigma^2 \tag{1.8}$$

Differentiating with respect to y_1 yields the first order condition

$$\frac{\partial w}{\partial y_1} = w_1 + \frac{1}{2} w_{\varepsilon\varepsilon 1} \sigma^2 = 0 \tag{1.9}$$

which implicitly defines the optimal choice y_1^*. Since at a maximum $w_{11} < 0$, the implicit function theorem in conjunction with (1.9) reveals that

$$\frac{\partial y_1^*}{\partial \sigma^2} \gtreqless 0 \text{ if } w_{\varepsilon\varepsilon 1} \gtreqless 0 \tag{1.10}$$

which is a particular case of Diamond and Stiglitz (1974) theorem 1. From (1.7), and using Young's theorem,

$$w_{\varepsilon\varepsilon 1} = w_{1\varepsilon\varepsilon} = U_{122} - U_{222} \tag{1.11}$$

Now, if U_{222} is positive and U_{122} is small or negative, $w_{1\varepsilon\varepsilon}$ will be negative and the choice of y_1 will be decreasing in σ^2. To interpret these third derivatives consider the rates of change of the marginal utilities of x_1 and x_2 with increases in the consumption of x_2 illustrated in Figure 1.1.

The signs of these derivatives are consistent with the proposition that x_2 is a necessity: its own marginal utility diminishes at a decreasing rate; and the marginal utility of x_1 increases[1/] with consumption of x_2 but at a diminishing rate. Thus consumption of x_2 in excess of some necessary level adds little to utility, and also the satisfaction resulting from the consumption of the other good increases with consumption of the necessity but at a decreasing rate.

Hence, this example shows that if nuclear power is a necessity, reductions in the uncertainties of nuclear technology should lead to a transfer of resources towards conventional generating capacity. The intuition behind this result is that given the definition of necessity, the consequences of having too little nuclear power is disastrous, therefore uncertainty leads the decision maker to aim high. As a result, reductions in uncertainty make it possible to reduce nuclear investment and still be as sure of reaching that target.

Additive uncertainty, as in example 1, is perhaps rather implausible. The degree of uncertainty is independent of any decision which the policy maker may take. Multiplicative uncertainty has more immediate economic interpretation since parameter estimates may be

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1/ The result is not affected if $U_{12} < 0$, i.e. the goods are substitutes.

$U_{22} < 0, \; U_{222} > 0$ $\qquad$ $U_{12} > 0, \; U_{122} < 0$

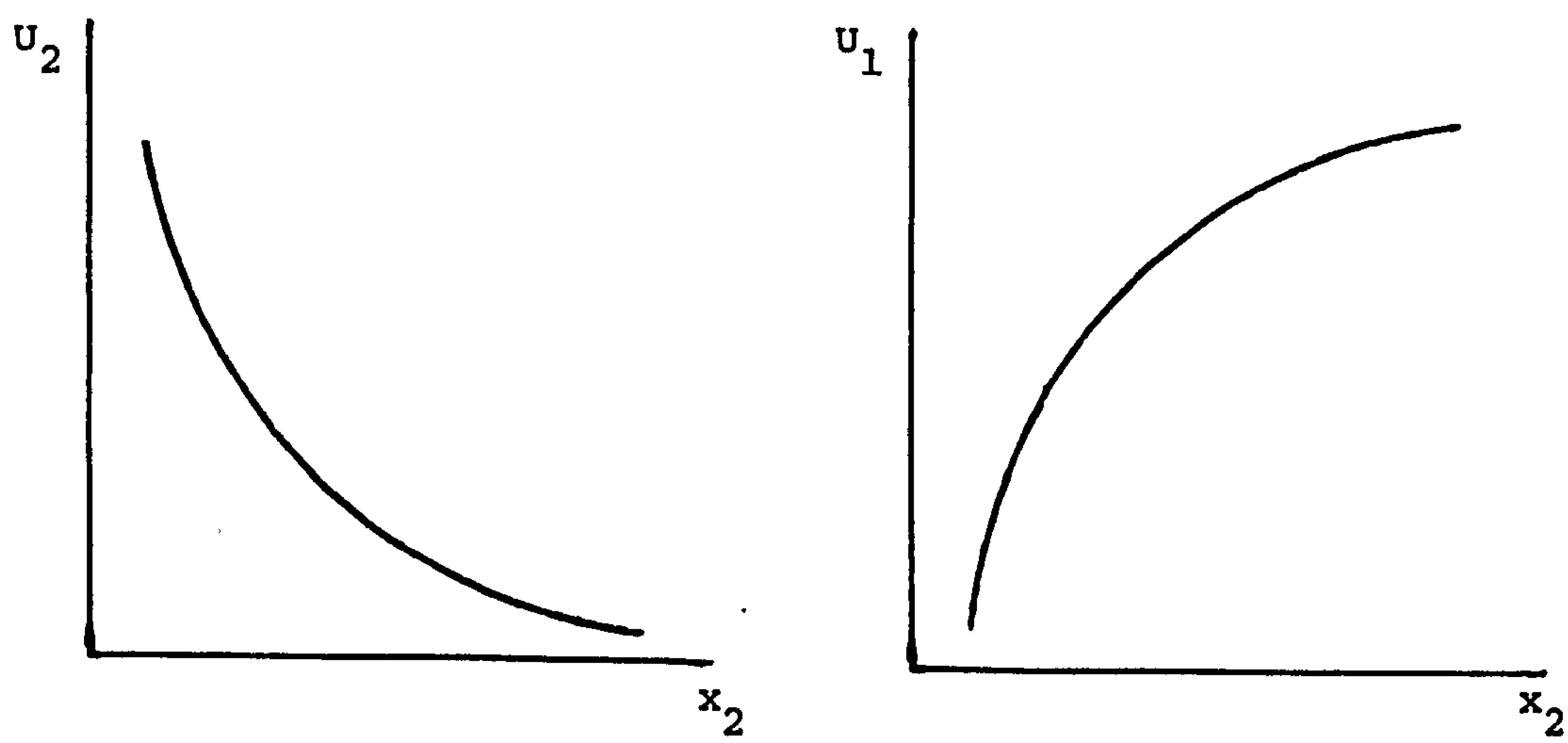


FIGURE 1.1 Third Derivatives, x_2 a necessity

the outcome of an uncertain process of econometric estimation. Also, with multiplicative uncertainty, responses to risk may be incorporated in a model with simple quadratic objectives. Such a model is described in example 2.

1.3.2 EXAMPLE 2

An inter-temporal public investment decision with risky returns, in a model with symmetric objectives, multiplicative uncertainty, and dynamic programming.

The model consists of two periods, in period one the decision maker chooses consumption and investment on the basis of the wealth endowment R. The rate of return on investment r is uncertain, and determines the amount available for consumption in the second period. A dynamic programming solution (see Intriligator, 1971, and Kamien and Schwartz, 1981), is presented (although the problem is simple enough for other methods to be applicable) since familiarity with this technique is required for the subsequent arguments. Using a Taylor series expansion around zero, and letting $U(0) = 0$, the symmetric objective function at time t may be written, suppressing the arguments of the functions on the right-hand side

$$U(x_t) = U_x(x_t) + \frac{1}{2} U_{xx}(x_t)^2 \tag{1.12}$$

Obviously, the optimal policy in period two is to consume everything

$$x_2 = (1 + r + \eta)(R - y_1) \tag{1.13}$$

where r is the rate of interest, and η is a stochastic disturbance with mean zero and variance s^2. By substitution into (1.12) the expectation of the maximised value for U_2 is

$$EU_2^* = U_x(1+r)(R-y_1) + \frac{1}{2}U_{xx}\left[(1+r)^2 + s^2\right](R-y_1)^2 \tag{1.14}$$

The period one problem is then to maximise present utility, taking account of the influence of current decisions on future opportunities.

$$\max_{y_1} J = EU_1 + \frac{1}{1+\rho} EU_2^* \tag{1.15}$$

where ρ is the rate of time preference. The first order condition for this problem implies

$$y_1 = \frac{U_x(r-\rho) + U_{xx}\left[(1+r)^2 + s^2\right]R}{U_{xx}\left[(1+\rho) + (1+r)^2 + s^2\right]} \tag{1.16}$$

The approximation entailed by (1.12) is only locally applicable, and certainly breaks down if $U_x + U_{xx}R > 0$ since this would mean a negative marginal utility of money. The expectation of y_2 is given by the budget constraint

$$Ey_2 = (1+r)(R-y_1) \tag{1.17}$$

Thus from (1.16) it is possible to show that

$$y_1 - Ey_2 > 0 \iff (\rho - r)\left[1 + \frac{(1+r)}{(2+r)} \frac{U_{xx}}{U_x} R\right] - \frac{s^2 R}{(2+r)} \frac{U_{xx}}{U_x} > 0 \qquad (1.18)$$

With no uncertainty, if the rate of time preference is greater than the rate of interest then more of the budget will be allocated to current rather than future consumption. The more uncertain is the rate of return on investment, and the larger is $-U_{xx}$, the greater is the tendency to plan to consume more at present than in the future. Dixit (1976) (example 9.1, pp.105-107) derives a similar result in a model with continuous time and perfect certainty: in his terminology the consumption path is increasing (decreasing) if the rate of interest is greater (less) than the rate of time preference. In the two period model presented here, the element of uncertainty about future returns tends to shift the allocation of funds away from investment in favour of more secure current consumption.

Examples 1 and 2 have attempted to show two particular ways in which precommitment, and the uncertainty connected with it, may affect optimal public sector decisions. In practice, policy-makers almost invariably have another instrument at their disposal: they may decide to delay making the decision. When information accumulates over time, this tool is particularly useful since delays lead to more accurate information and hence better decisions in the future.

1.4 OPTIMAL PROCRASTINATION

The potential for beneficial procrastination arises if the following conditions are fulfilled: firstly, there must be improvements in information over time; and secondly, the decision must have some consequences which may not be altered by subsequent policy revisions. Then, the benefits of delay will result from the ability to make a better informed decision, and the costs will be associated with the toleration of unsatisfactory conditions over the learning period.

Precommitment is essential to the argument. If there were no need to make decisions before all information was fully revealed, there could be no informational advantages to be gained from delay. Another important aspect is irreversibility. If decisions could be costlessly altered, then surely the best policy would be to update decisions sequentially as new information becomes available. However, restrictions on the ease with which decisions may be changed mean that mistakes may persist - good choices require good anticipation. For the time being, attention is concentrated on the most extreme form of irreversibility, where a single decision must be made once and for all. In this context the procrastinating government is deciding when to decide - just as the punter must decide when to place his bet.

The recent Sizewell enquiry is one of the few public sector decision-making procedures for which such extreme assumptions may be applicable. One of the many possible interpretations of the events is provided as an illustration. The date of the report of the enquiry may be regarded as the moment of decision: the recommendations made at that time having a decisive influence on the development of the PWR (Pressurised Water Reactor) programme. It seems unlikely that the enquiry could ever be repeated and therefore the decision may be regarded as once and for all. Throughout the previous decade, domestic and world energy markets were evolving, and information about nuclear technology was accumulating. It could be argued that there came a time when it was necessary to end the uncertainties about the UK nuclear strategy both because of the repercussions within the energy sector and the subsequent influence on other industries. At that time, the date of the enquiry was announced. The lag between the annoucement and the expected report of the enquiry may be interpreted as procrastination. The duration of that period effectively determined the amount of information which should be allowed to accumulate before the decision is made.

This example is clearly imperfect: the announcement of the enquiry and the proceedings themselves called forth new evidence which few could have anticipated. And the length of the enquiry, and the date of the report, clearly proved to be determined by events rather than being fixed in advance. However, the aspects of a once and for all choice, and the accumulation of information over time are the salient features for the following analysis.

These issues have been discussed in the industrial economics literature. In a game theory setting, firms who compete over prices are often regarded as precommiting themselves to a particular strategy by choice of capital. Commitment to a particular investment plan may tie the oligopolist to an aggressive pricing strategy which subsequently deters potential entrants to the market: "the role of an irrevocable commitment of investment in entry-deterrence is to alter the initial conditions of the post-entry game to the advantage of the established firm..." (Dixit, 1980). In a duopoly environment Benoit (1985) introduces research and development and finds that the potential for imitation may cause firms to delay innovations which would otherwise be profitable. "This is because the rival has the option of waiting to see if the innovation is a success before innovating, thus foregoing the risk of investment" (p. 105). The model below shares the property that anticipated changes in information are an important determinant of the optimal decision, but the analysis takes place in a passive random environment rather than a responsive game.

The model has three time periods. Period zero serves to establish a data base, consisting of N sub-periods, upon which the policy-maker's prior estimates are formed. At the end of period zero the policy-maker decides how long to procrastinate. This determines the number of sub-periods M in period one. At the beginning of period two, all available information is used to make the best choice of the instrument y which is fixed for the remaining T-N-M sub-periods. Until the beginning of period two, the instrument is assumed to be fixed at some non-zero value y_0. Under this assumption the procrastination decision determines how long the historically determined value of y_o is allowed to persist before the single optimal reform is undertaken.

The model is a univariate linear regression model with no intercept

$$x_t = \beta y_t + \varepsilon_t \tag{1.19}$$

The errors are known to have zero expectation ($E\varepsilon_t = 0$), and variance of σ^2 ($E\,\varepsilon_t^2 = \sigma^2$). Information about the parameter is assumed to be obtained by ordinary least squares. The formulae are presented allowing y to vary, since these expressions are required for chapter three. For this chapter y is subsequently assumed constant over the first two periods. At the beginning of period one least squares gives the following estimate of the parmeter and variance

$$\bar{\beta}_1 = \frac{\sum_{i=1}^{N} y_i x_i}{\sum_{i=1}^{N} y_i^2} \tag{1.20}$$

and

$$v(\beta_1) = \frac{\sigma^2}{\sum_{i=1}^{N} y_i^2} \tag{1.21}$$

The data set at the beginning of period two has an additional M observations. The least squares estimate of β on the basis of the pooled data set is

$$\bar{\beta}_2 = \frac{\sum_{i=1}^{N+M} y_i x_i}{\sum_{i=1}^{N+M} y_i^2} = \frac{\bar{\beta}_1 s^2 + \sum_{i=N+1}^{N+M} y_i x_i}{s^2 + \sum_{i=N+1}^{N+M} y_i^2} \qquad (1.22)$$

where

$$s^2 = \sum_{i=1}^{N} y_i^2$$

The estimate $\bar{\beta}_1$ is unbiased with variance $v(\beta_1)$, therefore the relationship between $\bar{\beta}_1$ and the true value of β may be written as

$$\beta = \bar{\beta}_1 + \nu . \qquad (1.23)$$

where ν is independent of ε_i, $i > N$, $E\nu = 0$, and $E\nu^2 = v(\beta_1)$. Using (1.23) in (1.22) the parameter estimate in period two may be written as

$$\bar{\beta}_2 = \bar{\beta}_1 + \frac{\sum_{i=N+1}^{N+M} (y_i^2 \nu + y_i \varepsilon_i)}{s^2 + \sum_{i=N+1}^{N+M} y_i^2} \qquad (1.24)$$

The variance of the estimate is

$$v(\beta_2) = \frac{\sigma^2}{s^2 + \sum_{i=N+1}^{N+M} y_i^2} \qquad (1.25)$$

This formulation makes clear the distinction between those elements which are known at the beginning of period one (namely $\bar{\beta}_1$ and s^2) and those which are not (namely ν, y_i, and ε_i where $i = N+1,\ldots,N+M$). The learning aspect of the problem concerns the influence of new data on the estimate of the parameter and its variance. The expectation of the parameter at the beginning of period two is therefore conditional upon the particular drawing of random variables. A considerable simplification is obtained by invoking the assumption that the instrument is fixed in periods zero and one. This reduces the random element of the right-hand side of (1.24) which may now be written

$$\bar{\beta}_2 = \bar{\beta}_1 + \frac{M\nu}{N+M} + \frac{\sum_{i=N+1}^{N+M} \varepsilon_i}{(N+M)y_o} \qquad (1.26)$$

This expression contains the random variables ν and $\sum \varepsilon_i$. These both depend or the parameters of the model. Therefore, to simplify subsequent analysis it is convenient to normalise these random variables as follows. Let

$$\nu = f\sigma/(\surd N y_o) \qquad (1.27)$$

and

$$\sum_{i=N+1}^{N+M} \varepsilon_i = g\sigma/M \qquad (1.28)$$

where f and g are standard normal variables. Substituting these expressions into (1.26) gives $\bar{\beta}_2$ as a function of two standard normal variables. A further simplification is possible because a weighted sum of standard normal variables is a constant times a third standard normal variable. Given constants a and b

$$af + bg = \sqrt{(a^2+b^2)}h \tag{1.29}$$

where h is the third standard normal variable. Using this rule in conjunction with the previous normalisation, the final expression for $\bar{\beta}_2$ is

$$\bar{\beta}_2 = \bar{\beta}_1 + \frac{\sqrt{M}\sigma h}{\sqrt{N}\sqrt{(N+M)}y_o} \tag{1.30}$$

Clearly, the fixity of y also simplifies the expressions for the variances, and the other parameter estimate.

$$v(\beta_2) = \frac{\sigma^2}{(N+M)y_o^2} \tag{1.31}$$

$$v(\beta_1) = \frac{\sigma^2}{Ny_o^2} \tag{1.32}$$

$$\bar{\beta}_1 = \frac{\sum_{i=1}^{N} x_i}{Ny_o} \tag{1.33}$$

Equations (1.30) to (1.33) are the final expressions representing the government's information in this model.

The governemnt has quadratic objectives defined on deviations from the most preferred value $\hat{x}$. In the final period the government minimises

$$E_2 \sum (x_t - \hat{x})^2 \tag{1.34}$$

where E_2 denotes expectations conditioned on information consisting of the parameter estimate $\bar{\beta}_2$ and its variance $v(\beta_2)$. Given this information they choose a single value y^* for the instrument. This value applies to each of the T-N-M sub-periods in the final period. The final-period objective is to minimise

$$E_2 \sum_{t=N+M+1}^{T} (x_t - \hat{x})^2 = E_2 \sum (\beta y + \varepsilon_t - \hat{x})^2 \tag{1.35}$$

This expression may be expanded as follows

$$E_2 \sum (x_t - \hat{x})^2 = E_2 \sum [(\beta - \bar{\beta}_2)y + \bar{\beta}_2 y - \hat{x} + \varepsilon_t]^2$$

$$= (T-N-M)([v(\beta_2) + \bar{\beta}_2^2]y^2 + \hat{x}^2 + \sigma^2 - 2\bar{\beta}_2 y \hat{x}) \tag{1.36}$$

Note that there is assumed to be no covariance between β and ε. Differentiation of (1.36) gives the first order condition for optimality of y

$$\frac{\partial}{\partial y} E_2 \sum (x_t - \hat{x})^2 = 2(T-N-M)([v(\beta_2) + \bar{\beta}_2^2]y - \bar{\beta}_2 \hat{x}) = 0 \tag{1.37}$$

This condition is satisfied by

$$y^{*} = \frac{\bar{\beta}_2 \hat{x}}{v(\beta_2)+\bar{\beta}_2^2} \tag{1.38}$$

Clearly, the second order condition for a minimum is also satisfied.

The optimal value function is derived by substituting (1.38) into (1.36)

$$\Sigma(x_t-\hat{x})^2 = (T-N-M)\left[\frac{\bar{\beta}_2^2\hat{x}^2}{v(\beta_2)+\bar{\beta}_2^2} + \hat{x}^2 + \sigma^2 - \frac{2\bar{\beta}_2^2\hat{x}^2}{v(\beta_2)+\bar{\beta}_2^2}\right]$$

$$= (T-N-M)\left[\frac{\hat{x}^2 v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2} + \sigma^2\right] \tag{1.39}$$

This expression is the minimised value of final-period losses given information available at the beginning of the final period.

The period one objective is to minimise

$$J = M[E_1(\beta y_o-x)^2+\sigma^2] + (T-N-M)\left[\hat{x}^2 E_1\frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2} + \sigma^2\right]$$

(1.40)

Note that expectations E_1 are conditioned on period one information. The first order condition is

$$\frac{\partial J}{\partial M} = E_1(\beta y_o - \hat{x})^2 - \hat{x}^2 E_1 \frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2}$$

$$+ (T-N-M)\hat{x}^2 E_1 \frac{\partial}{\partial M} \frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2} = 0 \qquad (1.41)$$

The benefits of procrastination in terms of improving the effectiveness of future policies are represented by the last term in (1.41). The expectation in this term is difficult to evaluate because of the random variables in the denominator. Taking derivatives through the expression yields

$$E_1 \frac{\partial}{\partial M} \frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2} = E_1 \frac{[\bar{\beta}_2^2 \frac{\partial}{\partial M} v(\beta_2) - 2v(\beta_2)\bar{\beta}_2 \frac{\partial \bar{\beta}_2}{\partial M}]}{[v(\beta_2) + \bar{\beta}_2^2]^2} \qquad (1.42)$$

The difficulty of taking expectations of this expression is apparent since, recalling that $\bar{\beta}_2$ is a linear function of the random variable h, the numerator has terms in h^2, and the denominator in h^4. This suggests that an analytic solution will not be forthcoming. Moreover, despite the intuitive appeal of the assertion that the expected value of final-period losses must fall with procrastination, no proof that (1.42) is negative is available.

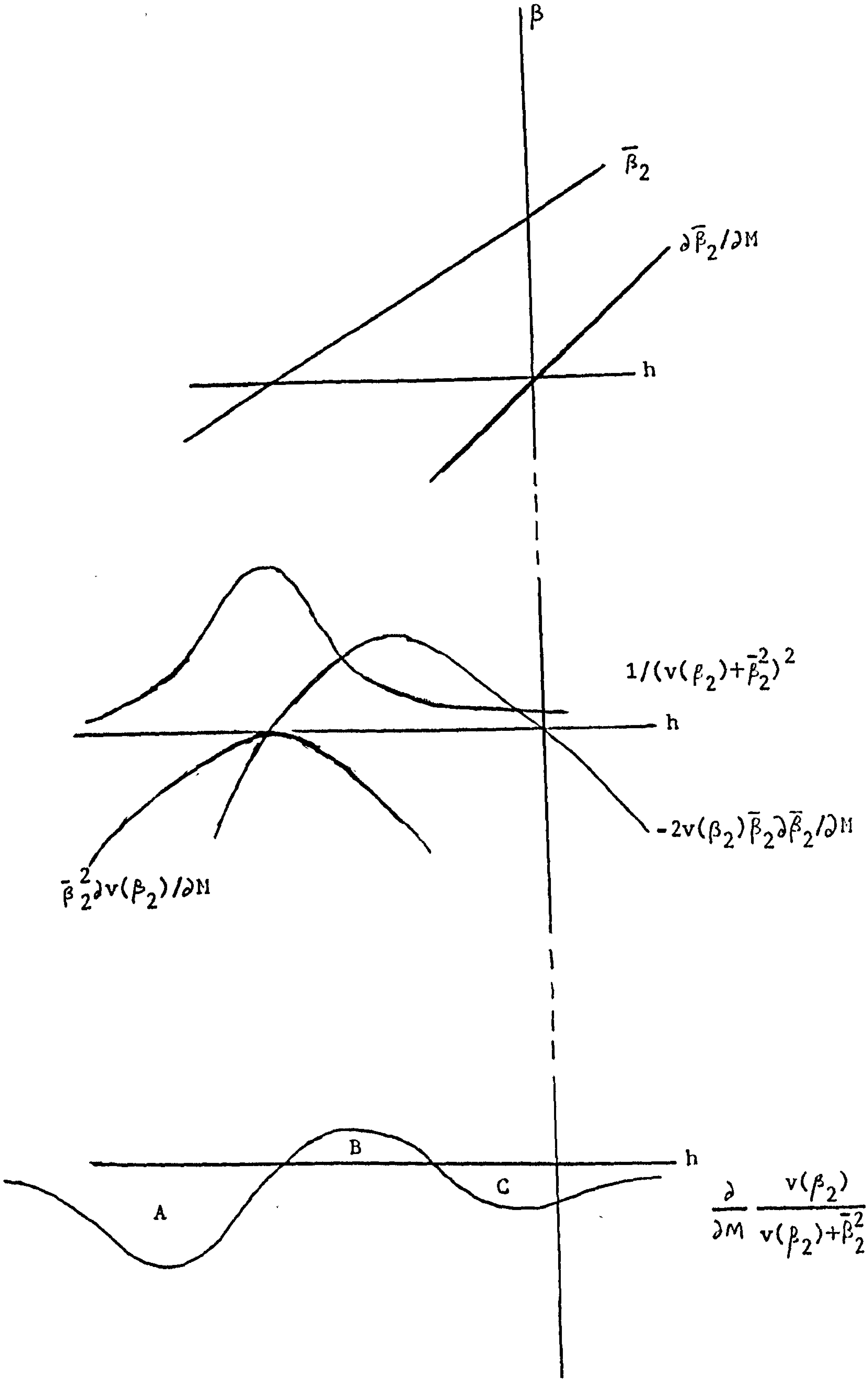
It is possible to illustrate the complexity of the problem with a sketch. Figure 1.2 shows the value of (1.42) for any given realisation of h. The top section uses (1.30) to sketch $\bar{\beta}_2$ and its derivative. Note that

$$\bar{\beta}_2 = 0 \Rightarrow h = -\frac{\bar{\beta}_1 \sqrt{N}\sqrt{(N+M)}y_o}{\sqrt{M}\sigma} \quad (1.43)$$

The middle section sketches the denominator, and the two terms in the numerator of (1.42). To confirm the shapes of these curves, recall from (1.31) that $v(\beta_2)$ is independent of h, and its derivative with respect to M is negative. The bottom section of the figure sketches the whole expression. The expectation of the expression is simply the weighted average of the three areas A, B, and C in the figure, the weights determined by the normal distribution for h. Since h is distributed around zero, . receives most weight, followed by B, and A the least. The sign of the expectation is not clear from this discussion. However, the sketch does make clear the following. The positive area B, in the final section of the figure, arises since there exist values of h which ensure that $\bar{\beta}_2 \partial\bar{\beta}_2/\partial M$ is negative. This means that the desirability of procrastination associated with reducing the variance of the parameter estimate may be partly or wholely offset by possible changes in the estimate itself.

Despite the complexity of the general problem, it is possible to describe the solution in certain

Figure 1.2


β
$\bar{\beta}_2$
$\partial\bar{\beta}_2/\partial M$
h
$1/(v(\beta_2)+\bar{\beta}_2^2)^2$
h
$-2v(\beta_2)\bar{\beta}_2\partial\bar{\beta}_2/\partial M$
$\bar{\beta}_2^2\partial v(\beta_2)/\partial M$
B
h
A
C
$\frac{\partial}{\partial M}\frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2}$

restricted cases. For this purpose it is convenient to re-write the first order condition as

$$\frac{\partial J}{\partial M} = E_1[(\beta y_o - \hat{x})^2 - (\beta y^* - \hat{x})^2] + (T-N-M)E_1\frac{\partial}{\partial M}(\beta y^* - \hat{x})^2 \tag{1.44}$$

It is immediately apparent (using the implicit function theorem) that since T only enters the last term, increasing T has a positive effect on the optimal value of M, provided that the derivative in this last term is negative.

A necessary condition for $M \neq 0$ to be optimal is that the last term in (1.44) is negative. Otherwise. the corner solution $M = 0$ would apply. Note that when $M = 0$, the first term on the right-hand side of (1.44) is non-negative. The first three of the special cases which follow indicate situations where no procrastination is desirable, and argue that under their particular assumptions the last term in (1.44) is zero.

Using (1.23) and noting that ν is independent of subsequent observations, the last term may be expanded as

$$E_1\frac{\partial}{\partial M}(\beta y^* - \hat{x})^2 = 2([\bar{\beta}_1^2 + v(\beta_1)]E_1 y^* \frac{\partial y^*}{\partial M} - \hat{x}\bar{\beta}_1 E_1 \frac{\partial y^*}{\partial M}) \tag{1.45}$$

Note that the envelope theo· .n does not apply because

y^* is optimal for $E_2(\beta y-x)^2$ not $E_1(\beta y-x)^2$.

CASE 1 T=N+M

Provided that the last term in (1.44) is bounded, this assumption clearly implies that the term is zero. This shows that if reform has not been undertaken by the last period it will certainly be undertaken then. The reason for delaying reform is that information, and hence the subsequent decision may be improved. By the time the last period has been reached, any considerations of the future are irrelevant. Therefore, at that moment the choice is simply between the single-period benefits of reforming and not reforming. Given that the reform would be optimal, the best decision is to reform. An alternative explanation of this result may be formulated in terms of option values. When the model has several time periods remaining, the decision to reform closes the option to make a reform at some subsequent date. The option has a positive value because information is expected to improve and so too is the effectiveness of the decision made no the basis of that improved information. However, at the last period time itself has closed any option to reform in the future, therefore the option value (or the returns to procrastination) have diminished to zero. Indeed Roberts and Weitzman (1981) show that this proposition applies to a much broader class of models.

CASE 2 Perfect Information

When $\sigma^2=0$, the parameter β is known with certainty, therefore the optimal choice of the instrument is $\hat{x}/\beta$. This decision is not affected by M, so the derivatives on the right-hand side of (1.45) are zero, and thus the last term in (1.44) is also zero.

CASE 3 Complete Uncertainty

When $\sigma^2=\infty$, the estimate of β at any time would have infinite variance. The optimal choice of y in the final period would then be $y^*=0$, which is independent of M, and the result is the same as for case 1.

CASE 4 Favourable Initial Conditions

When the historically determined value y_0 is quite close to the y which would be chosen if reform were undertaken immediatly, the cost of delaying reform is quite low. This means that, provided procrastination reduces expected losses in the future, some delay is desirable. In terms of the first order condition (1.44) setting M to zero would mean that the first term was zero and provided the second was negative, a non-zero M would be optimal. It is possible to identify the values for y_0 when this case arises by setting

$$y_o = \frac{\bar{\beta}_1 \hat{x}}{\bar{\beta}_1^2 + v(\beta_1)} \tag{1.46}$$

Substituting for $v(\beta_1)$ from (1.32)

$$y_o = \frac{\bar{\beta}_1 \hat{x}}{\bar{\beta}_1^2 + \sigma^2/Ny_o^2} \tag{1.47}$$

which yields a quadratic whose solution is

$$y_o = \frac{\bar{\beta}_1 \hat{x} \pm \sqrt{(\bar{\beta}_1^2 x^2 - 4\bar{\beta}_1^2 \sigma^2/N)}}{2\,\bar{\beta}_1^2} \tag{1.48}$$

If y_0 takes either of these values, immediate reform would not be desirable since it would simply implement the existing value for y.

These simple cases are implied by the general formulation of the problem in (1.44). To explore more fully the properties of the model with least squares estimation, the results of numerical simulations are presented. The objective function (1.40) is rewritten, expanding and taking expectations through the first term

$$J = M([\bar{\beta}_1^2+v(\beta_1)]y_o^2 - 2\bar{\beta}_1\hat{x}y_o+\sigma^2+\hat{x}^2) + (T-N-M)[\hat{x}^2E_1\frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2}+\sigma^2] \tag{1.49}$$

This expression was programmed along with the definitions of the estimates (equations (1.30)-(1.33)). The computer program then makes a number of drawings from a standard normal distribution and calculates the value of the objective function for each one. The expectation of the objective is simply the average of these values. The optimal value for M is identified by trying all integer values for M between 0 and T-N and choosing the best. Although this method is computationally slow, it does not require any derivatives, nor does it need to consider problems of local optima. The same random numbers were re-used at each iteration so that the merits of different possible M's were evaluated on the same drawing of random numbers. 120 random numbers were used, 60 being antithetics (that is, minus one times the other 60). The results are presented in tables 1.1-1.4). Each table contains the optimal integer value for M given the assumed parameter values.

Table 1.1 explores the different combinations of σ^2 and T. The limiting cases described above explain the following

i) the first colomn is zero because there is perfect certainty

ii) The first row is zero because T=N+M

And the comparative static effect of increasing T is

iii) for any value of σ^2, a higher T yields no less procrastination.

Reading across the rows of table 1.1 shows that for a given T, increasing σ^2 usually increases M at first, and then reduces it. When T=35 the greatest procrastination occurs when σ^2 lies between 2 and 3 (assuming that the results may be interpolated), and for other values of T maximum procrastination occurs at lower values of σ^2.

Table 1.2 experiments with combinations of y_0 and T. Once again, increasing T never reduces M (other things being equal). Reading across the rows of table 1.2 shows that in these examples for a given T, M has two local maxima. These occur when initial conditions, ie the value of y_0, are quite favourable. Note that for the parameter values underlying these examples

$$y_o = \frac{\hat{x}\bar{\beta}_1}{v(\beta_1) + \bar{\beta}_1^2} \quad \Rightarrow \quad y_o = 0.1 \quad \text{or } 1.72$$

Thus when y_0 takes either of these values the

Table 1.1 Optimal integer values for M.

Assumed Parameter Values:

$\overline{\beta}_1=1$, $y_0=7/4$, $\hat{x}=2$, $N=5$

		σ^2							
		0	1	2	3	4	5	6	7
T	5	0	0	0	0	0	0	0	0
	10	0	0	0	0	0	0	0	0
	15	0	2	1	1	0	0	0	0
	20	0	3	3	2	2	0	0	0
	25	0	4	4	4	3	3	2	0
	30	0	5	5	5	4	4	3	2
	35	0	5	6	6	5	5	4	3
	40	0	6	7	6	6	6	5	4
	45	0	7	7	7	7	6	6	5
	50	0	8	8	8	8	7	7	6

Table 1.2

$\bar{\beta}_1=1.1,\ \sigma^2=1,\ \hat{x}=2,\ N=5.$

y_0

		.1	.46	.82	1.18	1.54	1.90	2.26	2.62
	5	0	0	0	0	0	0	0	0
	10	3	0	0	0	0	0	0	0
	15	5	0	0	1	2	1	0	0
	20	6	0	2	3	3	2	0	0
T	25	8	4	3	4	4	2	0	0
	30	9	6	4	5	5	3	0	0
	35	11	7	5	5	6	4	0	0
	40	12	8	6	6	7	5	1	0
	45	13	10	7	7	8	5	1	0
	50	15	11	8	8	8	6	2	0

conditions are particularly favorable for procrastination. Note also that the higher values of y_0 are associated with less procrastination in these examples. This may be explained by the inverse relationship between y_0 and $v(\beta_1)$.

Table 1.3 shows examples of the interactions between y_0 and σ^2. As expected, when σ^2 is very small or very large, less procrastination is desirable regardless of the value of y_0. Note also that higher values of σ^2 reduce the value of y that would be chosen for immediate reform. Therefore the most favourable initial conditions entail a lower y_0, the higher is σ^2.

Table 1.4 allows $\bar{\beta}_1$, and y_0 to vary. The interacions between $\bar{\beta}_1$ and y_o seem to be quite complicated. For a given $\bar{\beta}_1$, M may increase or decrease with y_0; and for a given y_0, M may increase or decrease with $\bar{\beta}_1$.

Table 1.3

$\bar{\beta}_1=1.1$, $T=25$, $\hat{x}=2$, $N=5$

		y_0							
		.1	.46	.82	1.18	1.54	1.90	2.26	2.62
	0	0	0	0	0	0	0	0	0
	1	8	4	3	4	4	2	0	0
	2	8	10	6	6	5	2	0	0
	3	8	12	9	6	4	2	0	0
σ^2	4	8	13	11	7	4	1	0	0
	5	8	13	11	7	4	0	0	0
	6	8	13	11	6	3	0	0	0
	7	8	12	11	6	2	0	0	0
	8	8	11	11	6	0	0	0	0
	9	8	11	10	5	0	0	0	0

Table 1.4

$\sigma^2=1$, $T=25$, $\hat{x}=2$, $N=5$

y_0

$\bar{\beta}_1$	.1	.46	.82	1.18	1.54	1.90	2.26	2.62
.2	7	7	7	7	6	4	0	0
.4	7	8	7	4	3	2	2	2
.6	7	8	4	3	3	2	2	2
.8	8	7	3	3	3	3	3	1
1.0	8	5	3	3	4	3	1	0
1.2	8	4	3	4	4	1	0	0
1.4	8	4	4	5	3	0	0	0
1.6	8	4	4	6	0	0	0	0
1.8	8	4	5	4	0	0	0	0
2.0	8	4	6	2	0	0	0	0

1.5 CONCLUDING REMARKS

The possible connections and interactions between information and decisions are many and complex. The literature on choice under uncertainty has concentrated on the contribution of poor information to optimal choices. Two examples of public sector investments were presented under the heading of precommitment to show, by analysis of fairly standard models, the diversity of possible outcomes. This line of enquiry is pursued a little further in Chapters 5 and 6 where optimal taxation decisions are the focus of attention.

Just as information influences decisions, so too can decisions affect information. The procrastination model displayed this property in its simplest form: by delaying the decision more data may be brought to bear on the problem. In terms of the racing analogy, the punter who decides well in advance certainly frees himself from the worry of making a decision at the track; but the more dedicated gambler who waits until the last minute before placing a bet has available the most up to date information. Indeed it may be possible to do more than simply wait and see. The punter can buy the form book, or travel to the stables to see the horses in training. These 'active learning' strategies are considered in Chapter 3 with tax reform in developing countries in mind.

The once-and-for-all nature of the procrastination model is quite restrictive. In practice the consequences of public sector decisions tend to be measured in flows over time rather than discrete lumps. Also, the simple OLS information structure has the property that any uncertainty will eventually disappear. The alternative view that even the long run may be risky is used in the sequential decision making models of Chapter 2.

APPENDIX 1.1

Absolute and relative risk aversion are increasing in x if $U_{xxx} < 0$. From Deaton and Muellbauer (1980) pp.398-399 absolute and relative risk aversions are given by

$$RA = -\frac{U_{xx}(x)}{U_x(x)} \quad ; \quad RR = -\frac{xU_{xx}(x)}{U_x(x)}$$

Differentiating with respect to x yields, suppressing the argument

$$\frac{\partial RA}{\partial x} = -\left[\frac{U_{xxx}U_x - (U_{xx})^2}{(U_x)^2}\right]$$

$$\frac{\partial RR}{\partial x} = -\left[\frac{xU_{xxx}U_x + U_{xx}U_x - x(U_{xx})^2}{(U_x)^2}\right]$$

Since $U_x > 0$, $x > 0$, and $U_{xx} < 0$, a sufficient condition for both of these derivatives to be positive is $U_{xxx} \leq 0$.

CHAPTER TWO

THE TIMING OF POLICY REFORMS

CHAPTER TWO SUMMARY

The state of the economy is assumed to evolve according to a Markov process. The policy-maker must decide when to implement costly reforms, and what policies to implement. Two types of cost are distinguished.

When the accumulation of data (by means of a survey) is expensive, the policy-maker undertakes reforms at the date of every survey, and simply has to decide on the basis of currently available information, the optimal date for the next survey. The duration until this date increases with the survey costs, but either increases or decreases with uncertainty according to the expected effect on the future stream of net benefits.

When the reform itself is expensive, the policy-maker freely observes the evolution of the economy and must decide what values of the state variable would be sufficient to trigger the next reform. When the state variable remains within the dead band delimited by the trigger values, old policies continue. The comparative statics are i) the dead band expands with increasing reform costs, ii) the dead band shifts in the same direction as shifts in the government's priors and iii) if risk increases the dead band shifts and changes in width, the direction of these effects depending on the riskiness of the situation. This last result arises because of simultaneity between the choice of trigger values and the choice of the instrument.

2.1 Introduction

Keeping up to date is an expensive business. In a rapidly changing economic environment, policy decisions are rarely once-and-for-all; rather they may be regarded as one of a sequence of reforms. Recognition of the importance of the concepts of an evolving economy, and the sequential nature of decisions, raises the question addressed by this chapter: what criteria should be used to determine the data of the next policy reform?

The assertion that the economy is always in a state of change contradicts one of the fundamental premises of much economic theory. In existing dynamic models (such as Buiter and Miller, 1982) the concept of a long run steady state plays a crucial role. Difficulties in empirically identifying a long run steady state have recently been emphasised by Kelly (1985). Its existence, whether or not it may be identified in practice, implies that shocks, and imperfect information are relevant only to the short run: given sufficient time for all processes of adjustment to work through, the economy will surely settle again at some state where all economic parameters can be known with certainty. By expelling the idea of a long run steady state and replacing it with a state of perpetual flux, imperfect knowledge is built into the model as a durable rather than transient phenomenon. Although more data accumulates over time, older observations become less informative about the current state of the economy, and less useful as a guide for policy decisions.

Markov processes have two main advantages for the present purpose. Firstly, since current information is a sufficient statistic for these processes, expectations about future states of the economy are very simple. Secondly, these processes have the required effect of ruling out a perfectly informed steady state, because although the parameters of the process may be known with certainty, the information set required to forecast the state of the economy is always changing. Whether or not the Markov process gives a good approximation to the behaviour of economic variables is an empirical question beyond the scope of this thesis. However, there is some evidence in their favour. For example, Hall's (1978) econometric estimates of stochastic life cycle models support the proposition that consumption follows a random walk, unpredictable shifts in permanent income ringing the changes.

Whereas in a static economy decision-making may be modelling as once-and-for-all optimisation, in a changing environment a particular policy choice must be regarded as a temporary measure. Sequential decision-making is an essential ingredient of many established economic models: in super games for example, each player's action is simply one element in the whole stream of choices making up his strategy (see Friedman, 1971). The model in this chapter is perhaps more closely related to the tax reform literature. Tirole and Guesnerie (1981) concentrate on the sequential aspects of reform: the current reform affects future opportunities; option values, in the sense of Henry (1974a) arise; and it is not obvious that all possible sequences of improving reforms necessarily lead to optima. In order to concentrate on the timing of policy reforms, the models in this chapter do not rely on the structure of the tax reform problem, but are defined in terms of abstract states and instruments. As a consequence, the concept of amelioration used by Popper (1945), Guesnerie (1977) and Ahmad and Stern (1984) is lost: the government does not make a sequence of improving reforms, they re-optimise at each stage.

The issues involved in the timing of policy reforms are closely related to those of optimal stopping rules. Roberts and Weitzman (1981) discuss the sequential nature of the decision to continue or terminate a research programme. The decision to continue at any point in time is conditional on the ability to stop at some time in the future. They find that this tends to make continuation a more favourable choice "... the bias towards tentatively going ahead with a project is more pronounced as the variance of benefits is greater because the realisation of a stage removes more uncertainty and allows a better informed decision to be made" (p.1263). In a situation where the planning horizon is finite and only one policy reform may be made, the Roberts and Weitzman results are directly applicable: the single reform is analogous to the decision to stop the research project. However, in general the policy reform problem would allow any number of reforms in the future.

This chapter does not attempt the most general solution to the problem since the multi-period dynamic programming problem would be quite complicated. Instead, the intention is to describe the properties of the solution in greatly simplified cases. Clearly, it is the costs of re-optimisation which justify any delay in up-dating policies. Two sources of costs are examined. Firstly, it may be expensive to change the instrument. Administrative or bureaucratic costs are the immediate justification for this assumption. However in a macroeconomic context significant destabilisation may result from the adjustment of private sector behaviour, and the revisions of expectations, in response to policy changes. Secondly, data collection and processing costs could be the over-riding consideration. In developing countries, for example, tax reforms are frequently made on the basis of large-scale cross-section surveys which are extremely expensive to organise. The skilled labour tied up in such activities has obvious opportunity costs in terms of private sector production.

The general structure of the model is described in Section 2.2. The cost specification determines the kind of strategy for determining the timing of reforms. If data costs are important, then all relevant information is available at the time of the last survey, and the best strategy[1/] is to announce at that date the timing of the next reform. This option is detailed in Section 2.3. Section 2.4 examines reform costs in a two-period model where information accumulates freely over time. In this case the government re-evaluates the costs and benefits of reform at the end of the first period. This strategy may be described in terms of trigger values, indicating how far the economic situation would be

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1/ The optimality of this, and the trigger value strategy are discussed in Appendix 2.I.

allowed to deteriorate before policy reform costs were justified. Section 2.5 concludes.

2.2 The Model

To capture the evolution of the economy, assume that the state variable β follows a Markov process

$$\beta_t = \beta_{t-1} + \varepsilon_t \tag{2.1}$$

where

$$\varepsilon \sim D(\bar{\varepsilon},\sigma^2)$$

Beliefs about the future trajectory of the economy are incorporated in the distribution of ε. It is assumed to be characterised by mean and variance. This formulation allows a variety of distributions (including normal and uniform); and the non-zero mean would incorporate some trend in the ε's. For the present, however, it is convenient to assume no trend, $\bar{\varepsilon} = 0$. Thus the government's best guess about the values that β will take in the future is the most recent observation. However, the more time has elapsed since the last observation, the greater will be the variance of their estimate of β. This is illustrated in Figure 2.1. Observation takes place at time $t = 0$, and whatever the path of β up to that point, the expectation of β at any time in the future is β_0.

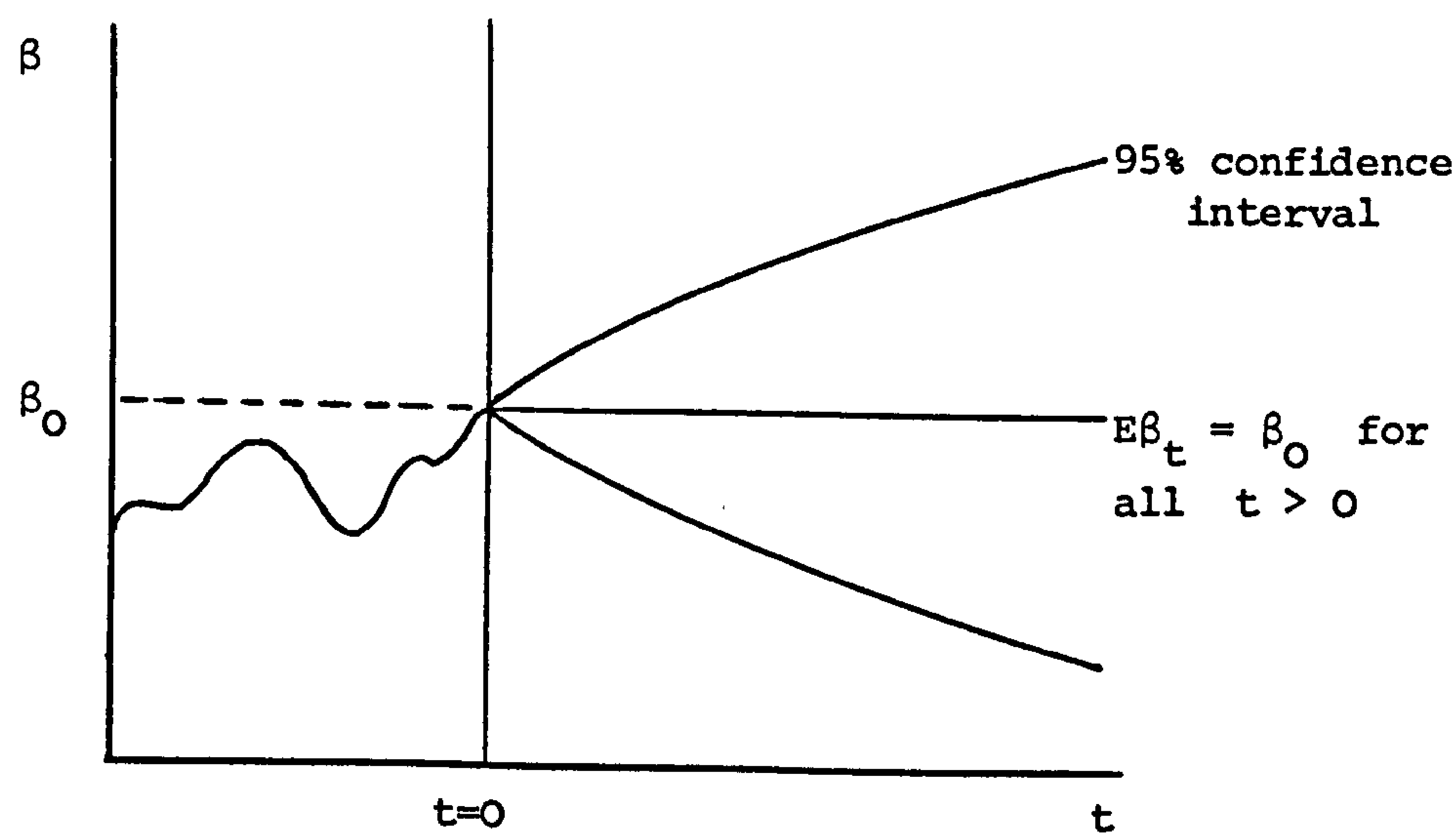


FIGURE 2.1 : Forecasting a Markov Process

To construct the confidence interval consider the variance of the estimate of β_t, assuming zero covariance

$$\begin{aligned} \operatorname{var} \bar{\beta}_t &= E(\beta_0 - \beta_t)^2 \\ &= E(\beta_0 - \beta_0 + \sum_{i=1}^{t} \varepsilon_i)^2 \\ &= t\sigma^2 \end{aligned} \tag{2.2}$$

The further into the future the forecast is applied, the greater is the uncertainty surrounding it.

The state variable describes the influence of the government's instrument y on the outcome (or target variable) x.

$$x_t = y_t \beta_t \tag{2.3}$$

Thus the expected outcome, conditional on observation β_0 is

$$Ex_t = y_t\, E(\beta_0 + \sum_{i=1}^{t} \varepsilon_t) = y_t \beta_0 \tag{2.4}$$

and its variance is

$$\begin{aligned} \operatorname{var} x_t &= E(x_t - Ex_t)^2 \\ &= E\left[y_t\beta_0 - y_t E(\beta_0 + \sum_{i=1}^{t} \varepsilon_t')\right]^2 \\ &= y_t^2 t\, \sigma^2 \end{aligned} \tag{2.5}$$

The government is assumed to have a valuation function defined on the difference between the outcome at any period and the most preferred outcome $\hat{x}$

$$U = U(x_t - \hat{x}) \tag{2.6}$$

The criterion for the evaluation of policy alternatives is then the future discounted flow of these payoffs.

Some additional notation is now introduced to draw out the sequential nature of policy choices. Assuming that a policy reform takes place at period 0, and remains in force until the next reform, M_0 periods later, the expected flow of benefits over the current reform period evaluated at time $t = i$ (where $0 < i < M_0$) and conditional on observation β_i, may be written

$$\underset{\beta_i}{E} \sum_{t=i}^{M_0} \delta^t U(x_t - \hat{x}) \equiv V \tag{2.7}$$

where δ is the discount factor, and V is condensed notation for the expected sum. The entire expected flow of net benefits is simply the discounted sum of benefits over each reform period less costs. This may be written

$$J(y_0, \beta) = V + \delta^{M_0} EJ(y_{M0}, \beta_{M0}) - \text{costs} \tag{2.8}$$

where the costs are specified below. Notice that the expectations implicit in J are evaluated on the basis of the most recent information, β_i, and that the future benefits EJ depend on the policies and information applicable to subsequent reforms.

The decision of whether or not to reform entails comparison between the benefits obtained by re-optimising, with those which would arise if old decisions were allowed to persist in states to which they were not perfectly suited. Let $y(\beta)$ be the value of y which maximises J for any given β, and y_0^* be that value which maximises J if β_0 is the state of the economy. Then it is fairly easy to establish that the functions $J(y_0^*,\beta)$ and $J(y(\beta),\beta)$ satisfy an envelope property defined by the following.

i) When β_0 arises the payoffs are the same when y is fixed as y_0^* and when y is allowed to vary optimally. This is true because by definition

$y(\beta)$ maximises J given β

and y_0^* maximises J given β_0

Therefore $y(\beta_0) = y_0^*$ (2.9)

and $J(y_0^*,\beta_0) = J(y(\beta_0),\beta_0)$ (2.10)

ii) The payoff holding y fixed at y_0^* is less than that obtained by allowing y to vary optimally when $\beta \neq \beta_0$. Let $\tilde{\beta}$ be any value for β other than β_0, then

$$J(y(\tilde{\beta}),\tilde{\beta}) > J(\tilde{y},\tilde{\beta}) \tag{2.11}$$

where $\tilde{y}$ is any value for y other than $y(\tilde{\beta})$

Therefore

$$J(y(\tilde{\beta}),\tilde{\beta}) > J(y_0^*,\tilde{\beta}) \tag{2.12}$$

iii) Given any change from β_0 to $\tilde{\beta}$, the change in $J(y(\beta),\beta)$ will be greater than (less than) the change in $J(y_0^*,\beta)$ as $\tilde{\beta}$ is greater than (less than) β_0. This may be written

$$\frac{\Delta J(y(\beta),\beta)}{\Delta\beta} \gtrless \frac{\Delta J(y_0^*,\beta)}{\Delta\beta} \iff \tilde{\beta} \gtrless \beta_0 \tag{2.13}$$

which may be expanded as

$$\frac{J(y(\beta_0),\beta_0)-J(y(\tilde{\beta}),\tilde{\beta})}{\beta_0-\tilde{\beta}} \gtrless \frac{J(y_0^*,\beta_0)-J(y_0^*,\tilde{\beta})}{\beta_0-\tilde{\beta}} \tag{2.14}$$

$$\iff \tilde{\beta} \gtrless \beta_0$$

The equivalence is then apparent from i) and ii) which allow a cancellation, and give the sign of $J(y(\tilde{\beta}),\tilde{\beta}) - J(y_0^*,\tilde{\beta})$. Thus

$$\frac{J(y(\tilde{\beta}),\tilde{\beta})}{\tilde{\beta} - \beta_0} \gtrless \frac{J(y_0^*,\tilde{\beta})}{\tilde{\beta} - \beta_0} \iff \tilde{\beta} \gtrless \beta_0 \tag{2.15}$$

If, in addition, the functions are continuous and concave, and $U(x-\hat{x})$ is always decreasing in deviations from $\hat{x}$, then Proposition 2.2 may be restated in terms of the derivatives of $J(y(\beta),\beta)$ and $J(y_0^*,\beta)$. The argument then concentrates on the relative magnitudes

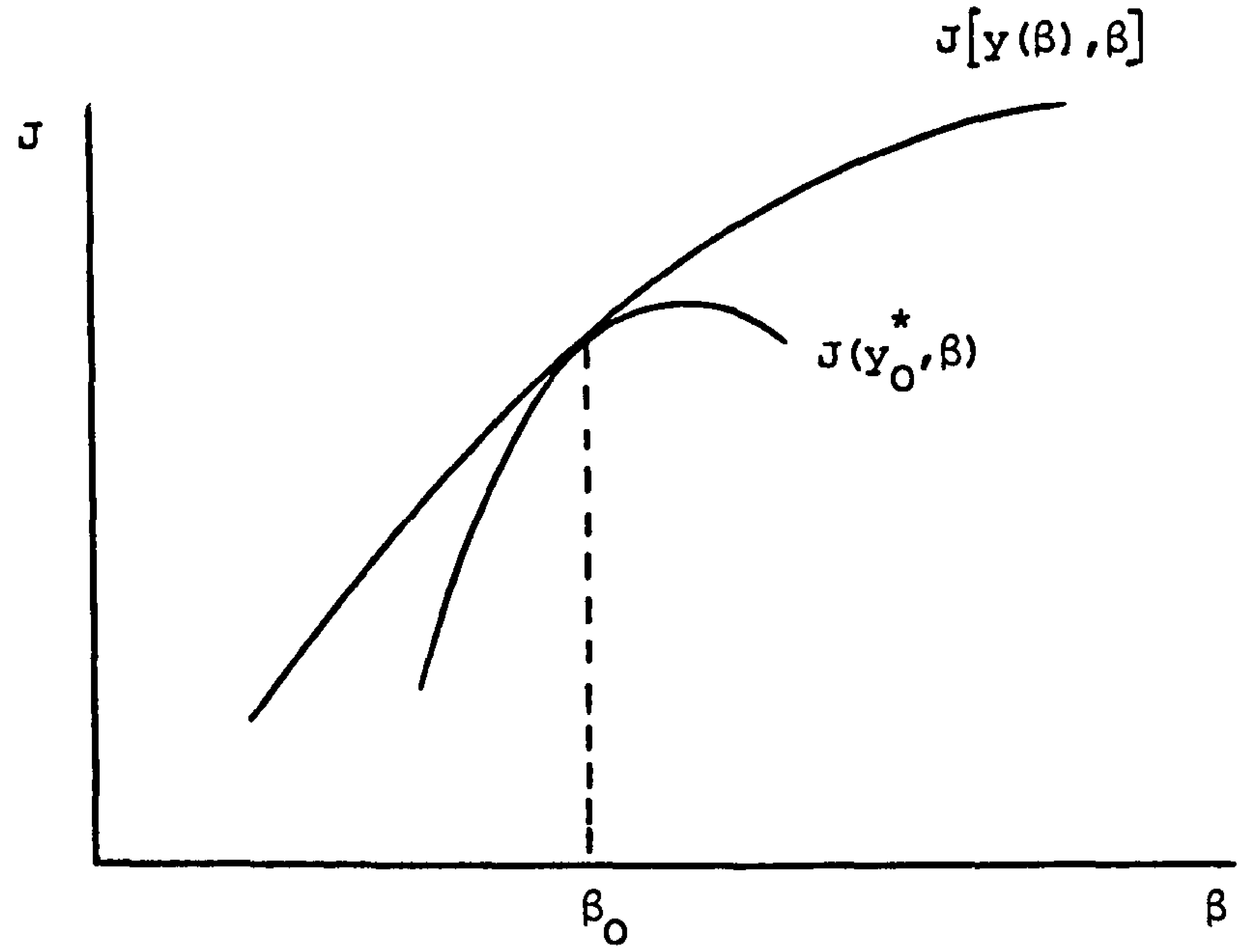


FIGURE 2.2 : Given policy choice y_O^*, the ideal state is β_O

of the second derivatives of the functions. Figure 2.2 is a sketch of the functions under these assumptions. Notice also that $J(y(\beta),\beta)$ could equally well be downward sloping, the important characteristic being concavity.

Changes in β will be particularly important in Section 2.4, where this variable triggers policy reforms. However the case to be considered first has costly information acquisition and so new information can only be obtained at the dates of subsequent reforms.

2.3 Costly Data

The simplest representation of data costs is to define c as the fixed costs of conducting a survey. At the date of the last reform this cost must have already been incurred to provide the information set β_O. Therefore future costs will apply at the dates of subsequent reforms. In addition, this section assumes that the state of the economy returns to β_O at the date of each reform. This assumption is not intended to have a literal interpretation; no reform, however powerful, could truly set the clock back. Instead it is intended as an approximation which asserts that for the purpose of policy decisions, the future path of the economy is perceived as being similar after each reform. This assumption results in a considerable simplification because it means that the infinite future time path of the economy may be broken down into identical subsections. The valuation function may then be written in the form

$$J = E\left[\sum_{t=0}^{M-1} \delta^t U(\hat{x}-y_t\beta_t) - \delta^M c + \delta^M J)\right] \tag{2.16}$$

The problem is then similar to one where a single reform must be undertaken in a finite horizon. (2.16) may be rearranged as

$$J = \frac{1}{1-\delta^M} \; E\left[\sum_{t=0}^{M-1} \delta^t \; U(\hat{x}-y_t\beta_t) \; - \; \delta^M c\right] \qquad (2.17)$$

Although M and y are re-optimised at subsequent reforms, the problem faced at those future dates is always identical to today's problem. Therefore re-optimisation in the future simply amounts to repetition of the current decision.

In this simplified model, an increase in data costs cannot bring forward the optimal date of the next reform. The proof relies on two elements : firstly data costs are discounted from the date of the next reform, therefore reducing M would increase the present value of data costs; and secondly, because subsequent decisions are merely repetitions, there can be no interactions between future reform periods. This may be summarised as follows:

<u>Proposition 2.1</u>

<u>An increase in data costs delays the optimal date of the next reform</u>

Proof

Let M be the optimal reform date. Then, the payoff (2.17) may be written in the form

$$J(\beta,M) = B(M) - D(M)c \qquad (2.18)$$

where $B(M) = \frac{1}{1-\delta^M} E \sum_{t=0}^{M-1} \delta^t U(\hat{x} - y_t \beta_t)$

and $D(M) = \frac{\delta^M}{1-\delta^M}$

Optimality of M requires

$$B' - D'c = 0 \qquad (2.19)$$

where B' and D' are derivatives with respect to M. This first order condition implicitly defines the relationship between c and M. The implicit function theorem implies

$$\frac{dM}{dc} = \frac{D'}{B'' - D''c} \qquad (2.20)$$

The second order condition for optimality of M is

$$B'' - D''c < 0 \qquad (2.21)$$

Thus dM/dc has the opposite sign to D'. Clearly, $D(M)$ is decreasing in M : delaying reform reduces the present value of discounted costs. Therefore

$$\frac{dM}{dc} > 0 \qquad (2.22)$$

□

It would be possible to envisage situations where this result did not hold. For example, the policy maker might be faced with an array of alternative reforms, and a fixed budget constraint. Then the allocation of expenditure between possible reforms would be analogous to the consumer's decisions about expenditure on different goods, and the income effect could dominate the effect of an increase in the cost of a given reform. This effect relies on the composition of the Slutsky matrix (which contains the derivatives of the compensated demand for good i with respect to the price of good j) and clearly cannot apply in the one-good model specified here. However, the effect of changes in uncertainty are not so clearcut, even in the simple one-dimensional model.

An obvious difficulty is that of dimensionality. The decision-maker would generally be allowed to choose both the type and date of reform. With two independent choice variables, changes in risk could cause complicated interactions between these choices. The overall effect on the reform date would depend on the effect of risk on the optimal policy, and also the effect of the optimal policy on the reform date. To proceed any further it is necessary to make the simplifying assumption that the instrument y is determined by a rule which is independent of the degree of risk. This means that the influence of risk on the next reform date may be calculated as if y were fixed.

Let M be the planned reform date which maximises expected benefits given the initial degree of risk. Then the realised value of the objective function could be no greater if the reform were delayed by an amount i. This may be written

$$J(M) - J(M+i) \geq 0 \qquad (2.23)$$

If an increase in risk increases the value of (2.23) then it cannot mean that $M + i$ is a preferable reform date to M. Using (2.17) it is possible to write the derivative of (2.23) with respect to σ^2 as

$$\frac{\partial}{\partial\sigma^2}\left[J(M) - J(M+i)\right] = \frac{1}{1-\delta^M}\frac{\partial}{\partial\sigma^2} E \sum_{t=0}^{M-1} \delta^t U_t$$

$$- \frac{1}{1+\delta^{M+i}}\frac{\partial}{\partial\sigma^2} E \sum_{t=0}^{M+i-1} \delta^t U_t$$

$$= \left[\frac{1}{1-\delta^M} - \frac{1}{1-\delta^{M+i}}\right]\frac{\partial}{\partial\sigma^2} E \sum_{t=0}^{M-l} \delta^t U_t$$

$$- \frac{1}{1-\delta^{M+i}}\frac{\partial}{\partial\sigma^2} E \sum_{t=M+1}^{M+i-l} \delta^t U_t \qquad (2.24)$$

having defined $U_t = U(x-y_t\hat{\beta}_t)$. The derivatives of U with respect to σ^2 are negative by risk aversion, therefore the first term is negative $(\delta^{M+i} < \delta^M)$ and the second is positive.

In either of the limiting cases, $\delta = 0$ or $\delta = 1$, the first term in (2.24) is zero and so it cannot be optimal to delay reform in response to an increase in risk. However, there is no general result for $0 < \delta < 1$. The second term shows that delay may be harmful because it allows risks to persist which could be avoided by reform. The first term shows that delay may be desirable because it delays the arrival of what are now riskier future benefits.

However, in the simplest case of a quadratic valuation function a clearcut result is possible. The simplest formulation is to set y to unity and $\hat{x}$ to zero and assume that each time a reform is undertaken β is reset to zero. This formulation gives

$$J = \frac{1}{1-\delta^M} \left\{ \sum_{t=0}^{M-1} \delta^t t\sigma^2 - \delta^M c \right\} \tag{2.25}$$

Notice that this function is homogenous of degree one in σ^2 and c. This means that an equal proportional increase in σ^2 and c would have no effect on the optimal choice of M. However, proposition 2.1 established that increasing c alone increases M, therefore increasing σ^2 alone must reduce it. This argument may be summarised as follows

<u>Proposition 2.2</u>

<u>With quadratic objectives, the optimal reform date is brought forward as uncertainty increases</u>

Proof

The derivative of (2.25) may be written

$$\frac{\partial J}{\partial M} = \sigma^2 \frac{\partial}{\partial M} \left\{ \frac{\Sigma \sigma^t t}{1-\delta^M} \right\} - c \frac{\partial}{\partial M} \left\{ \frac{\delta^M}{1-\delta^M} \right\} = 0 \tag{2.26}$$

i) Assuming that the second order condition is fulfilled, the implicit function implies that the effect of a parameter change on M has the same sign as its effect on $\partial J/\partial M$.

ii) $\partial J/\partial M$ is obviously homogeneous of degree zero in c and σ^2. Therefore Euler's theorem implies

$$\frac{\partial}{\partial c}\frac{\partial J}{\partial M}\,c + \frac{\partial}{\partial \sigma^2}\frac{\partial J}{\partial M}\,\sigma^2 = 0 \tag{2.27}$$

iii) Proposition 2.1 established that the first term is positive (assuming c and σ^2 are positive), therefore the second is negative

$$\frac{dM}{d\sigma^2} < 0 \tag{2.28}$$

$\square$

The properties of the optimal decision rules in this model result from the assumption that data collection is expensive. Information arrives in discrete bundles on payment of c, and no new information can be brought to bear on the government's decisions until the next survey has been undertaken. Under the alternative specification, that data flows freely, decision rules must take into account optimal adaptations to the changing environment.

2.4 Costly Reform

The model of Section 2.3 must be revised to examine reform costs. The analysis is simplified by restricting attention to quadratic objectives, and a uniform distribution for ε. Also, it is useful to distinguish between extremely risky and slightly risky situations. The most important modification is, however, that the government receives new information every period about the evolution of the economy to date. On the basis of that information they must decide both whether to make a costly policy reform, and what reform to implement should any be desirable. The decision rule - reform or no reform - may be described in terms of trigger values. The government uses the new information to re-evaluate their prospects every period. When the expected future benefits, or the state variable, becomes unacceptably poor, the reform is triggered.

Assume that at the date of the last reform, time $t = 0$, the government chooses the value of the instrument y to maximise the discounted flow of future net benefits conditional on observation β_0. At some subsequent time t, if a reform were carried out, optimisation would be on the basis of the most up-to-date information β_t. The desirability of the reform depends on the outcome of a cost benefit analysis. The net present benefit of the reform at time t is

$$NPB_t = J(y_t^*, \beta_t) - J(y_0^*, \beta_t) - k \tag{2.29}$$

where $J(y_t^*, \beta_t)$ and $J(y_0^*, \beta_t)$ are the flows of benefits resulting from reform and no reform respectively, both evaluated on the basis of observation β_t, and k is the fixed cost of reform. Clearly, the reform would be undertaken at time t if it yielded a positive net benefit. When the net present benefit equals zero the government is indifferent between making the reform and persevering with the old policies. The values of β for which the net present benefit of reform is zero are the trigger values, and they are implicitly defined by setting equation (2.29) to zero. Assuming that there are two solutions for the trigger values, it is possible to sketch the problem as in Figure 2.3.

At time $t = 0$ a reform takes place and trigger values $\bar{\beta}$ and $\underline{\beta}$ are chosen. When the state of the economy reaches one of these values at $t = M_0$, another reform takes place and new critical values are chosen. There is no reason to believe that in general the dead band would be a horizontal rectangle of the type sketched in the figure.

Roberts and Weitzman (1981), for example, find that to justify the continuation of a research project, the prospects of continuation must become more favourable over time. That is, the lower trigger value slopes upwards over time. However, in support of the conjectured trigger values in the figure, it is possible to argue that since there are no option values (the current reform places no restrictions on subsequent reforms), and the variance of the Markov process is known and fixed, the factors determining the choice of trigger values in this model are unlikely to change over time. When, in addition, the model is restricted to two periods, such considerations are irrelevant because the trigger values are then applicable to only that moment in time when the decision about reform is made, and the compexities of determining the dead band as a function of time are eliminated.

By using the implicit function theorem in conjunction with (2.12) it is possible to make some judgements about the effect of reform costs on the trigger values.

$$\frac{d\beta_t}{dk} = - \frac{\partial NPB/\partial k}{\partial NPB/\partial \beta_t} \qquad (2.30)$$

Assuming for the time being that the decision rule determining y is independent of k, the numerator is negative. Therefore, the direction of influence has the sign of the denominator

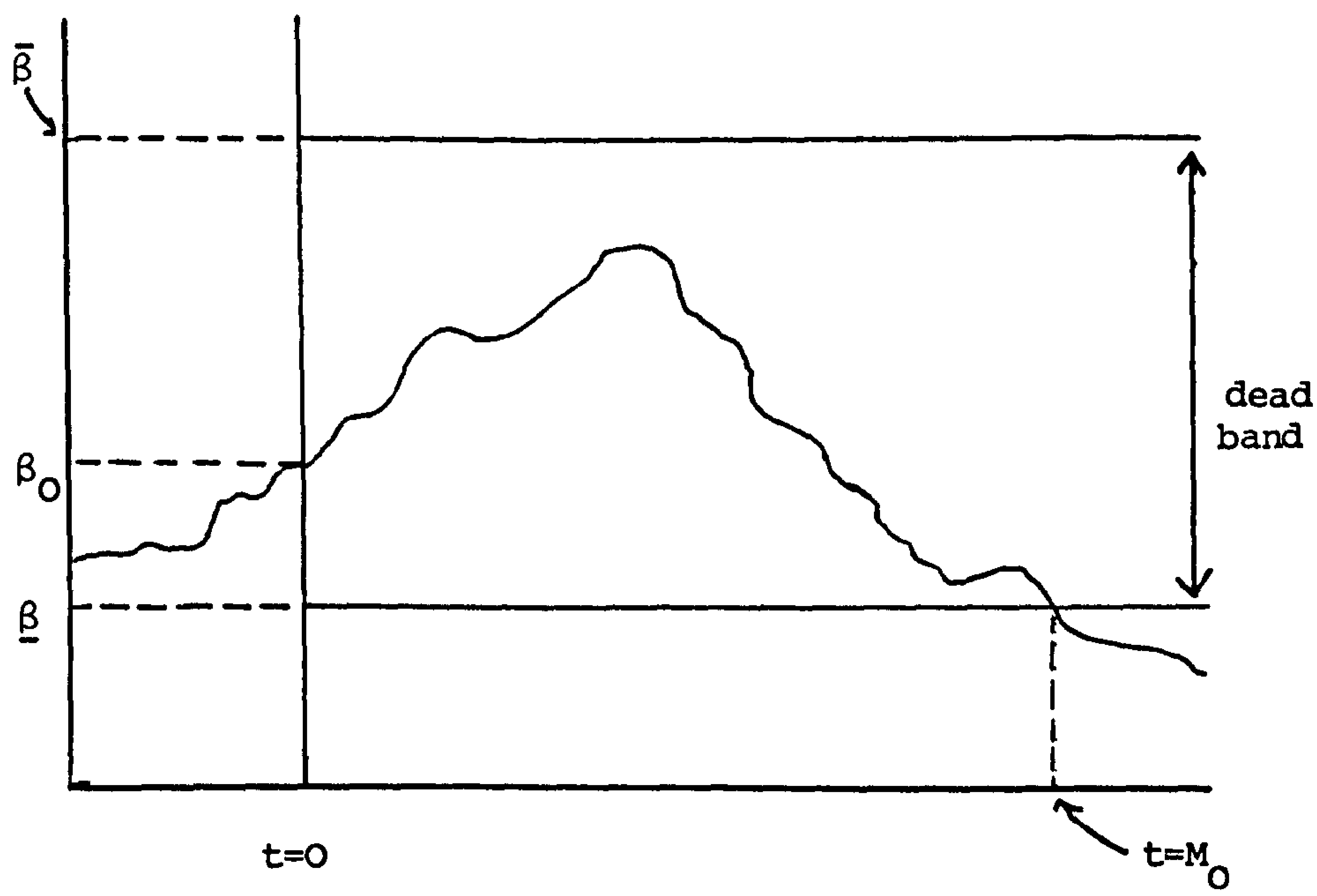


FIGURE 2.3 : The time path of β, trigger values, and the dead band

$$\frac{\partial NPB}{\partial \beta_t} = \frac{\partial J(y_t^*, \beta_t)}{\partial y_t^*} \frac{\partial y_t^*}{\partial \beta_t} + \frac{\partial J(y_t^*, \beta_t)}{\partial \beta_t} - \frac{\partial J(y_O^*, \beta_t)}{\partial \beta_t} \tag{2.31}$$

Optimality of y_t^* requires that the first term is zero. The relative magnitudes of the second two terms are determined by the envelope property defined in Section 2.2. Since y_O^* is fixed, changes in β have a stronger influence on $J(y_O^*, \beta_t)$ than on $J(y_t^*, \beta_t)$. Given also that y_O^* was optimal for β_O, the sign of the sum of the last two terms is the same as the sign of $\beta_t - \beta_O$.

$$\text{sign}\left\{\frac{\partial J(y_t^*, \beta_t)}{\partial \beta_t} - \frac{\partial J(y_O^*, \beta_t)}{\partial \beta_t}\right\} = \text{sign}\left\{\beta_t - \beta_O\right\} \tag{2.32}$$

Therefore the critical value of β_t increases if it is above β_O and falls if it is below.

$$\frac{\partial \bar{\beta}}{\partial k} > 0 \quad ; \quad \frac{\partial \underline{\beta}}{\partial k} < 0$$

As might be expected, higher reform costs would tend to reduce the desired frequency of reform, and this would be achieved by increasing the width of the dead band.

This argument is closely related to the proof of proposition 2.1 in the previous section that the reform date increases with data costs. A similar proof may be constructed here under the assumption that $\bar{\beta}$ and $\underline{\beta}$ are symmetric. This means that an increase in the width of the dead band b increases $\bar{\beta}$ and reduces $\underline{\beta}$. Under this simplifying assumption it is possible to write (2.17) in the form

$$J(\beta,M) = B(b) - D(b)c \tag{2.33}$$

where $B(b)$ is the expected benefits accruing until the expected reform date, and $D(b)$ is the expected discount factor applying to costs. Then proposition 2.1 may be restated in terms of the dead band as follows:

Proposition 2.3

The dead band width increases with reform costs

The proof is a trivial extension of the proof to proposition 2.1. The only additional argument required is that $D(b)$ is decreasing in b. This is obvious since an increase in the dead band width will make it less likely that a reform will have been undertaken by any given date. With discounting this means that the expected discount factors applying to future reform costs will fall.

However, this argument excludes the influence of k on decisions about y. In general, reform costs, and their influence on the trigger values would affect this choice. For example: the knowledge that reform will be undertaken if the state of the economy deteriorates beyond a certain level may affect the policy-maker's attitude to risk, and may enable him to pursue policies which would otherwise have been undesirable. To examine this issue, and also the more complicated influence of uncertainty, it is necessary to make some more assumptions, and examine the determination of y.

The solution for the optimal y is by means of dynamic programming. The current choice of y affects the trigger values which in turn influence the arrival dates and magnitudes of future costs and benefits. To simplify the analysis, it is convenient to make some more restrictive assumptions. Firstly, the model is restricted to two time periods, the second of which is intended to represent the entire future of the economy. By lumping together all future decisions into one period, a considerable simplification is gained, but connections between current choices and allocation within future periods is ruled out. Secondly a quadratic form is imposed for the evaluation function. Finally, a uniform distribution is used to represent beliefs about the errors in the Markov process determining the evolution of the economy. Before proceeding to the analysis it may be helpful to explain in some detail the importance of these assumptions for the optimal decision.

Since the objective function is of the quadratic form

$$U = - (y_t\beta_t - \hat{x})^2 \tag{2.34}$$

and the policy maker can choose y_t after β_t has been observed, the best policy in period one is simply to set $y_1 = \hat{x}/\beta_1$. This would ensure zero losses in period one, however the policy revision can only be justified if the cost of reform k was less than the losses associated with continuing under the old policy y_0. Indifference between the reform and the status quo occurs when

$$(y_0\beta_1 - \hat{x})^2 = k \tag{2.35}$$

The trigger values are derived by rearranging this equation as follows

$$\bar{\beta} = \frac{\hat{x} + \sqrt{k}}{y_0} \quad ; \quad \underline{\beta} = \frac{\hat{x} - \sqrt{k}}{y_0} \tag{2.36}$$

The width of the dead band is simply the difference between the two trigger values

$$\bar{\beta} - \underline{\beta} = \frac{2\sqrt{k}}{y_0} \tag{2.37}$$

The following properties of the trigger values emerge. Holding y_0 constant

i) Increasing k reduces $\underline{\beta}$ and increases $\bar{\beta}$ (as a consequence the dead band expands). As will become apparent, there are additional effects via the influence of k on y_0.

ii) Increasing the target $\hat{x}$ shifts both trigger values upwards but does not affect the dead band width.

iii) Increasing y_0 reduces both trigger values and reduces the dead band width.

In the dynamic programming solution to the period zero problem choice of y_0 affects the trigger values which in turn affect the payoff in the final period. The period zero criterion is

$$EU_0 = -(y_0\beta_0 - \hat{x})^2 + \delta\left[-\Pr(\text{reform})k + \Pr(\text{no reform})EU_1(y_0)\right] \tag{2.38}$$

The first term is the period zero benefits, and the second term, which includes the discount factor δ, includes the influence of y_0 on subsequent opportunities. Reform costs, and zero utility (omitted from the expression) are incurred in states where the reform is optimal, and benefits accruing to the persistence of y_0 occur otherwise.

Under the assumption of a uniform distribution for ε, beliefs about β in the final period may also be represented by a similar distribution. This is apparent from (2.1). With this distribution there is nothing to be gained by assuming that the expectation of ε is zero. The distribution of future β's may, without loss of tractability, be centred around some point other than β_0. Given the interval defined by the trigger values, and the interval between the upper and lower limits of the uniform distribution, four cases are possible. These are 1) the trigger values lie within the distribution, 2) the two intervals overlap but neither lies completely within the other 3) the distribution lies entirely within the dead band 4) the distribution lies entirely outside the bead band. The last two cases are of little interest because 3) never allows reform and 4) always requires reform. Therefore only the first two possibilities are considered.

2.4.1 CASE 1 : Very Risky Situations

This is called a very risky situation because a broad range of β's are equally likely. This is illustrated in Figure 2.4, where the distribution of β is over the interval $[L,H]$. When uncertainty is very great this distribution is stretched out; and it is therefore plausible to argue that under extreme risk, both of the trigger values for β will lie within the interval $[L,H]$. If this is the case, a small increase in uncertainty, other things being equal, reduces the likelihood of observing a β between the trigger values; density is shifted out of the dead band and the no-reform outcome becomes more likely. As a consequence, the decision maker would be more confident about aiming at the current period target, knowing that the chance that the current decision would be required to serve in less favourable future conditions was reduced. In terms of the quadratic example, a bigger spread for β would allow the government to aim higher, i.e. closer to the target. This reverses the usual comparative-static result. In a single-period model the policy maker would hedge against multiplicative uncertainty by reducing the value of the instrument. However, in the dynamic model, the possibility of subsequent reforms allows the instrument to be increased.

The probability of reform may in general be represented as the chance that β_1 falls outside $[\underline{\beta},\bar{\beta}]$; when both trigger values lie within the range of the distribution, this may be written as

$$\Pr(\text{reform}) = \frac{(H-L) - (\bar{\beta} - \underline{\beta})}{H-L} \tag{2.39}$$

and no reform takes place if β is inside the dead band

$$\Pr(\text{no reform}) = \frac{\bar{\beta} - \underline{\beta}}{H-L} \tag{2.40}$$

Substituting the probabilities into the period zero objective function (2.38) yields

$$EU_0 = -(y_0\beta_0-\hat{x})^2 - \delta\left\{\left[\frac{(H-L)-(\bar{\beta}-\underline{\beta})}{H-L}\right]k + \int_{\underline{\beta}}^{\bar{\beta}} (y_0\beta_0-\hat{x})^2 f(\beta_1)\,d\beta_1\right\}, \tag{2.41}$$

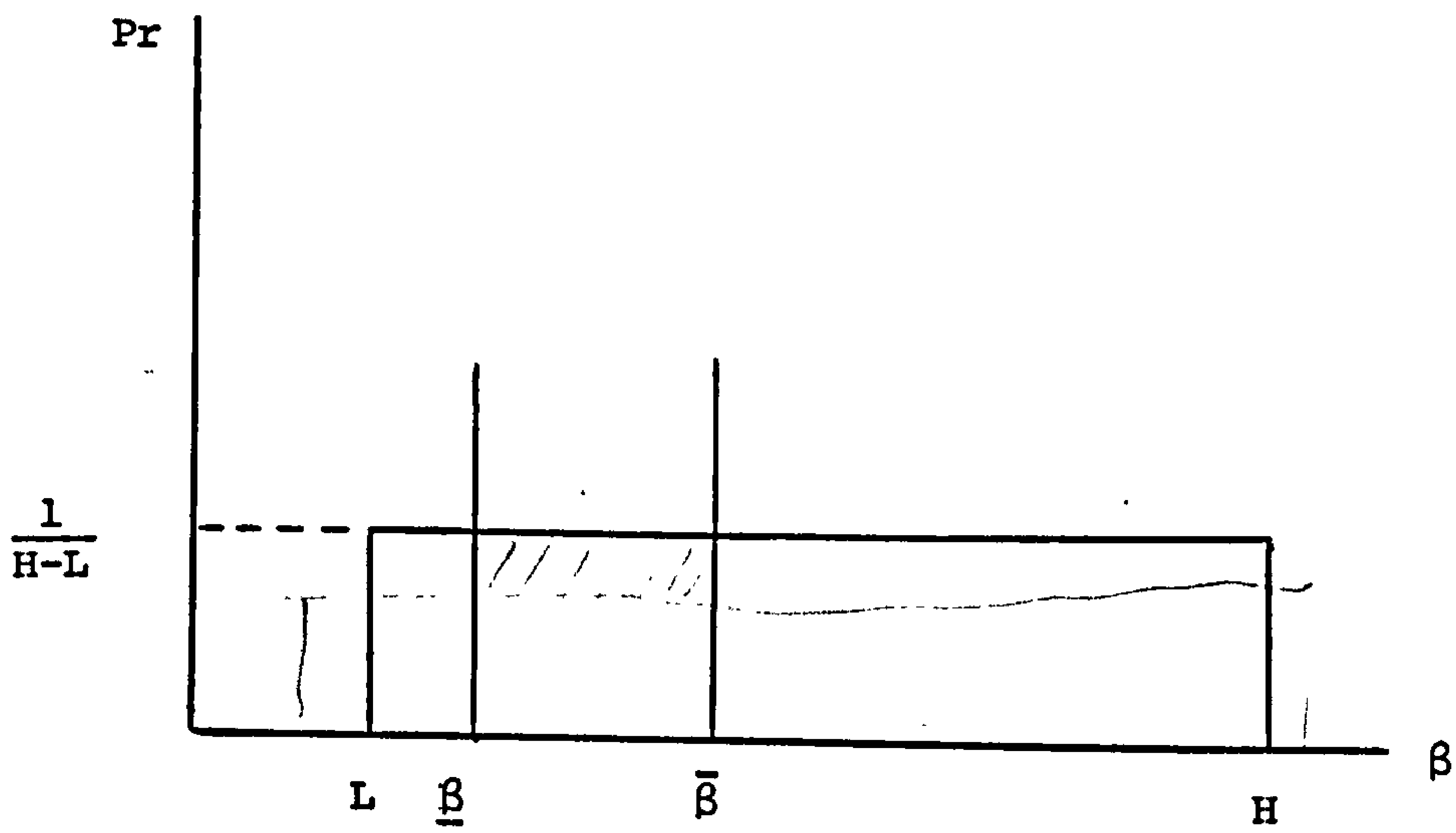


FIGURE 2.4 : A very risky distribution of β

Notice that the probability of no reform is included in the limits of the integral in the last term; expectations are taken only over those values of β within the dead band. The final step is to evaluate the integral, using the expressions for $\bar{\beta}$ and $\underline{\beta}$ from (2.36)

$$\int_{\underline{\beta}}^{\bar{\beta}} (y_0 \beta_0 - \hat{x})^2 f(\beta_1) d\beta_1 = \frac{1}{(H-L)} \frac{2}{3} \frac{k\sqrt{k}}{y_0} \tag{2.42}$$

Substituting this into the objective function (2.41) and using again the expressions for the trigger values (2.36)

$$EU_0 = -(y_0\beta_0 - \hat{x})^2 + \frac{\delta}{H-L} \left[-(H-L)k + \frac{4}{3} \frac{k\sqrt{k}}{y_0} \right] \tag{2.43}$$

The optimal choice of y_0, given its influence on the subsequent trigger values requires [1/]

$$\frac{EU_0}{y_0} = -2(y_0\beta_0 - \hat{x})\beta_0 - \frac{\delta}{H-L} \frac{4}{3} \frac{k\sqrt{k}}{y_0^2} = 0 \tag{2.44}$$

The implicit function theorem then reveals the following

i) $dy_0^*/dk < 0$

A higher reform cost leads the government to aim lower. More costly reform means that it is more likely that y_0 will be required to continue in period one, therefore it is reduced to hedge against uncertainty about what state will arise.

1/ The second order condition for a maximum requires

$$4\delta k\sqrt{k} < 3\beta_0^2 y_0^3 (H-L)$$

ii) $dy_0^*/d(H-L) > 0$

H-L is a measure of the spread of the uniform distribution, and may be interpreted as the degree of risk. More risk means that, other things being equal, y_0 is less likely to continue in period one, therefore it may be increased. The standard result that y^* decreases with uncertainty is reversed by the introduction of the trigger values.

Comparative static effects of exogenous variables on the trigger values may now be derived from (2.36) and (2.37)

$$\frac{d\bar{\beta}}{dk} = \frac{1}{2}\,\frac{k^{-\frac{1}{2}}}{y_0} - \frac{\hat{x} + \sqrt{k}}{y_0^2}\,\frac{dy_0}{dk} > 0$$

$$\frac{d\underline{\beta}}{dk} = -\frac{1}{2}\,\frac{k^{-\frac{1}{2}}}{y_0} - \frac{\hat{x} - \sqrt{k}}{y_0^2}\,\frac{dy_0}{dk} \gtrless ?$$

$$\frac{d(\bar{\beta}-\underline{\beta})}{dk} = \frac{k^{-\frac{1}{2}}}{y_0} - \frac{2\sqrt{k}}{y_0^2}\,\frac{dy_0}{dk} > 0$$

Increasing k increases the width of the dead band, through its effect on the numerator of (2.18). But it also reduces y_0 which appears in the denominator, and therefore shifts the dead band upwards. The sum of these two effects means that the upper trigger level increases but the lower could move in either direction depending on the relative magnitude of shift and expansion of the dead band.

The effects of uncertainty are quite straightforward

$$\frac{d\bar{\beta}}{d(H-L)} = -\frac{(\hat{x}+\sqrt{k})}{y_O^2}\frac{dy_O}{d(H-L)} < 0$$

$$\frac{d\underline{\beta}}{d(H-L)} = -\frac{(\hat{x}-\sqrt{k})}{y_O^2}\frac{dy_O}{d(H-L)} < 0 \quad \text{if} \quad \hat{x} > \sqrt{k}$$

More uncertainty increases y_O and shifts both trigger values downwards. Since they both move down by the same proportion the width of the dead band also shrinks.

$$\frac{d(\bar{\beta}-\underline{\beta})}{d(H-L)} = -\frac{2\sqrt{k}}{y_O^2}\frac{dy_O}{d(H-L)} < 0$$

Numerical simulations are presented to illustrate the analytic results. Figure 2.5 depicts the solution to the first order condition (2.26) for the chosen parameter values. The lines are downward sloping since, in agreement with the comparative statics, greater reform costs reduce the optimal value of y_O. The lowest of the lines represents a less risky situation: other things being equal, a higher value for the upper end of the distribution of β means greater spread (although the spread is not mean-preserving) and hence greater risk. Therefore the difference between the lines shows that for given reform costs, more risk means a higher y_O.

Figure 2.6 shows the results of the simulations for the trigger values themselves. Higher reform costs, as predicted, increase the width of the dead band. More uncertainty shifts the trigger values downwards, reducing the width of the dead band.

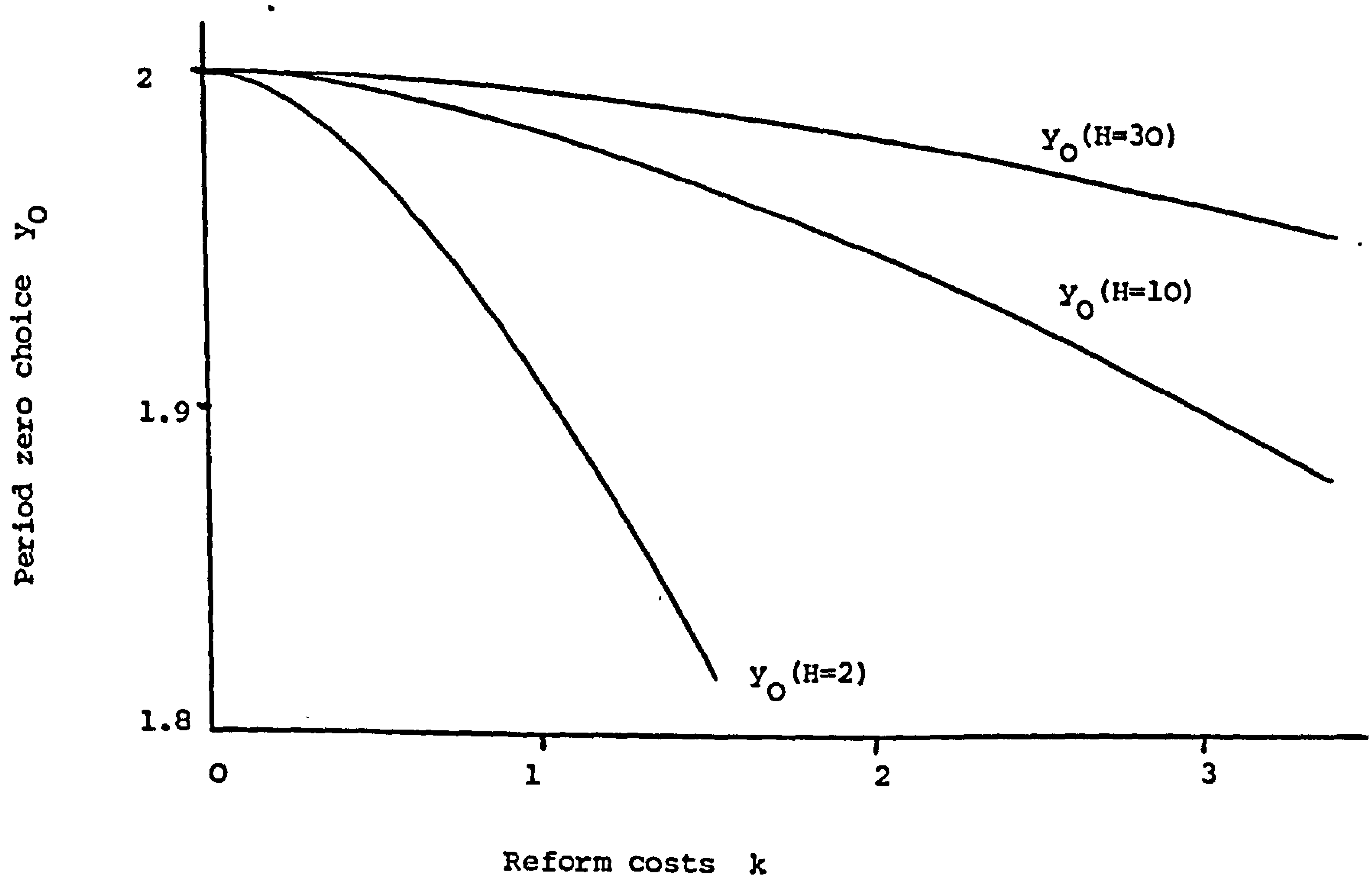


FIGURE 2.5 : Choice of y_0, given trigger values in period one.
Simulation with parameters $\beta_0 = 1$, $\hat{x} = 2$, $L = 0$

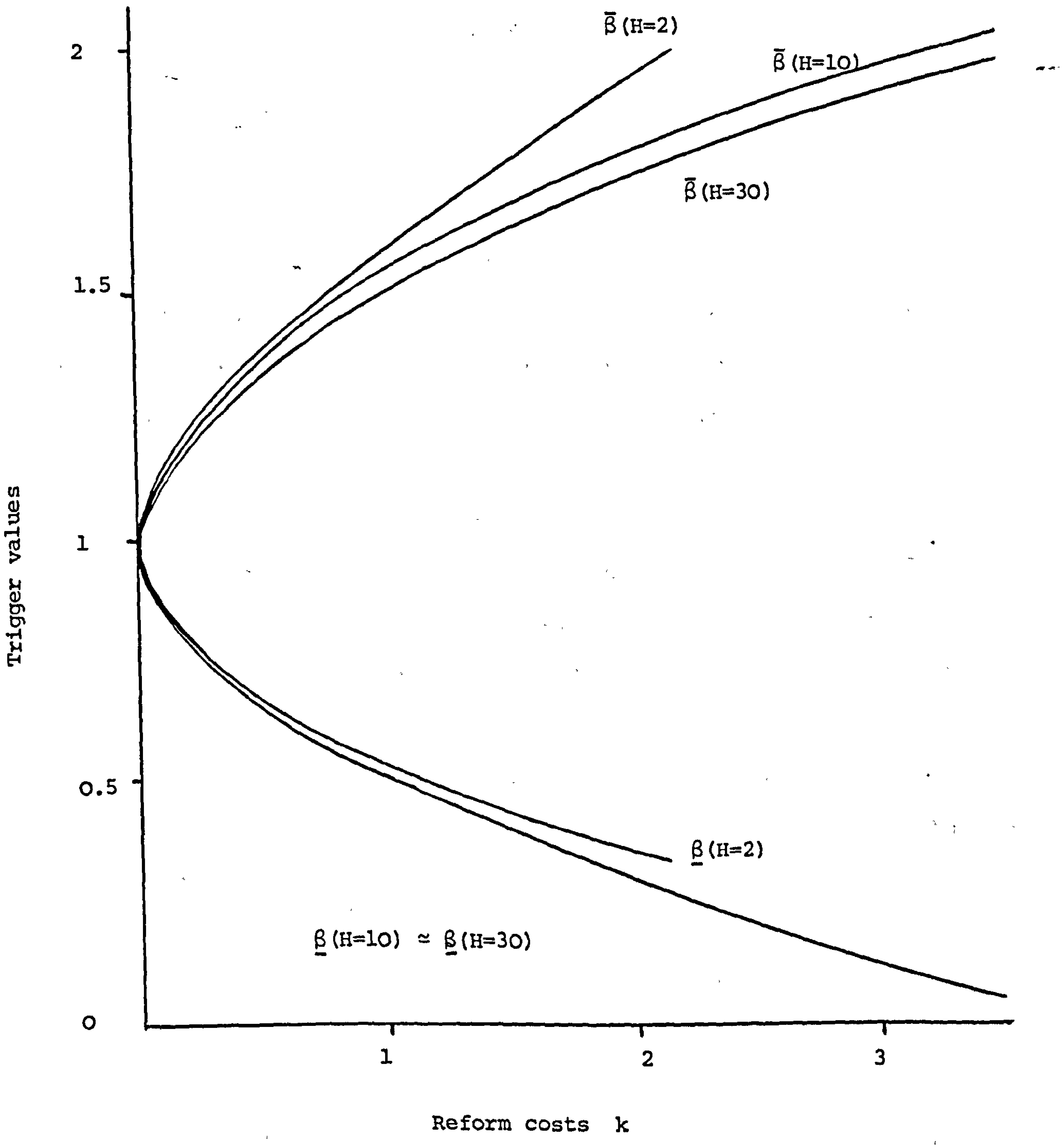


FIGURE 2.6 : Trigger Values : Simulation with Parameter Values: $\beta_0 = 1$, $\hat{x} = 2$, $L = 0$

2.4.2. CASE 2 : Slightly Risky Situations

A situation of slight risk is defined as one where the dead band lies partly but not completely within the range of the distribution of β. Figure 2.7 illustrates the case where β is uniformly distributed over a narrow interval which does not include both trigger values. In this case a slight increase in the spread of the distribution, holding its mean constant, shifts density into the dead band, making the no-reform option more likely. Now, such increases in risk would lead the government to place more weight on the possibility that current decisions would be allowed to persist. In the quadratic example they would aim lower.

Attention is restricted to the case where $\bar{\beta} < L \leq \underline{\beta}$ and $\bar{\beta} < H$ (illustrated in the figure) although the arguments apply equally well to the other similar case where $L < \underline{\beta}$ and $\underline{\beta} \leq H < \bar{\beta}$. The possibility that β can never become small enough to trigger the reform affects the possibilities of reform and no reform, and it is necessary to reconsider the analysis presented in case one accordingly. The period zero objective, with the revised probabilities is now

$$EU_0 = -(y_0\beta_0-\hat{x})^2 - \delta\left\{ \frac{H-\bar{\beta}}{H-L}\ k + \int_L^{\bar{\beta}} (y_0\beta_1-\hat{x})^2\ f(\beta_1)d\beta_1 \right\} \qquad (2.45)$$

Comparing (2.45) with (2.41) notice that the probability of reform is now independent of $\underline{\beta}$ since this value is known to be infeasible. Also, the lower bound on the integral in the last term now coincides with the lower bound of the probability distribution rather than the lower trigger level.

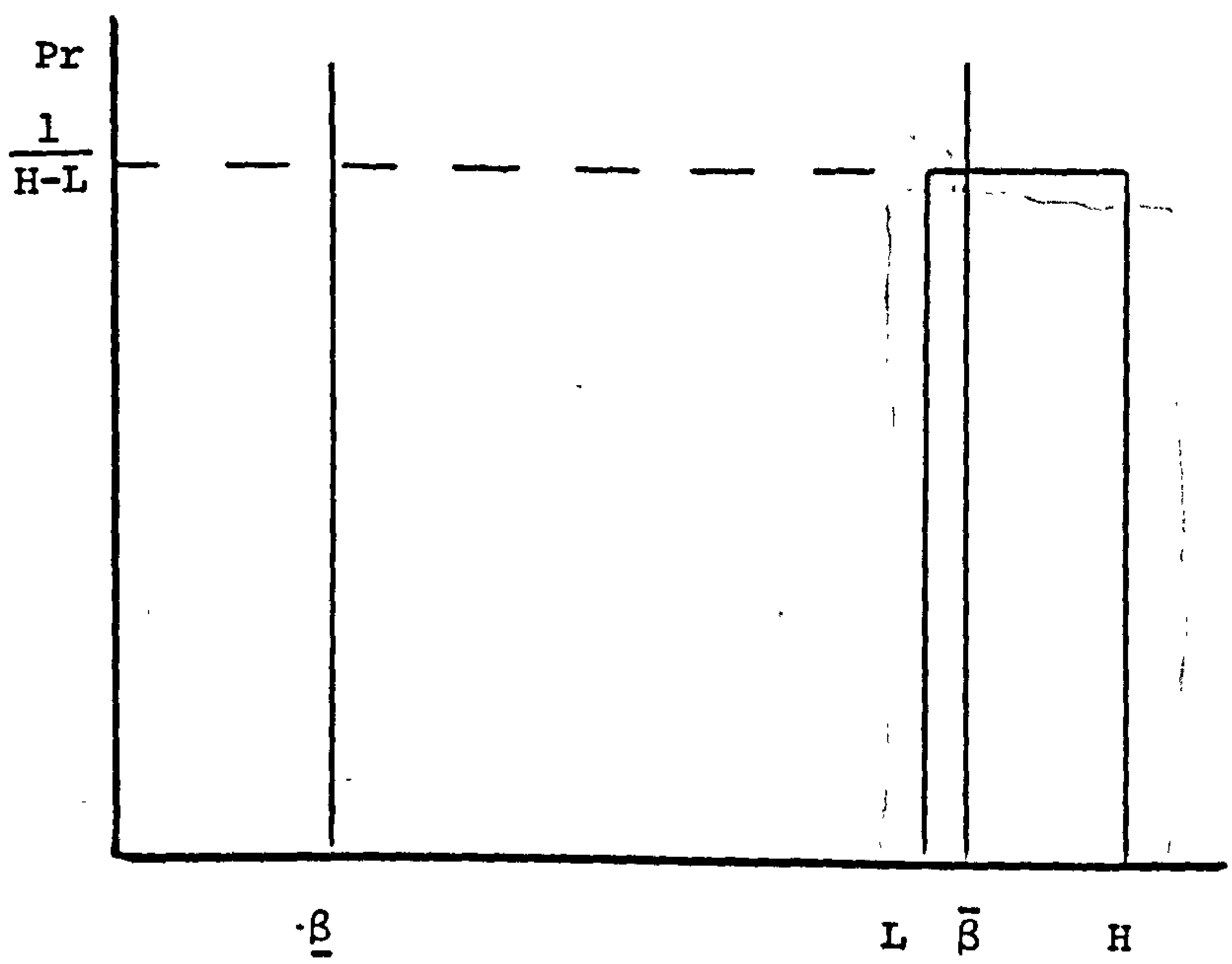


FIGURE 2.7 : A slightly risky distribution of β

The first order condition[1/] for optimality of y_O is

$$\frac{\partial EU}{\partial y_O} = -2(y_O\beta_O-\hat{x})\beta_O - \frac{2\delta}{H-L}\int_L^{\bar{\beta}} (y_O\beta_1-\hat{x})\beta_1 d\beta_1 = 0 \tag{2.46}$$

From this equation it is possible to derive the comparative static effects of changes in risk, and costs on the optimal values of y_O, and hence on the trigger values. Consider first changes in risk. A mean-preserving spread of the distribution may be represented as an increase in H accompanied by an equal reduction in L. Therefore differentiation of (2.46) with respect to risk, amounts to differentiating with respect to H, and L, setting $\partial H = -\partial L$. Therefore the effects of risk on y_O, given that the second order condition is also satisfied, has the opposite sign to

$$\frac{\partial^2 EU}{\partial y_O \partial L} = \frac{2\delta}{H-L}\left[(y_O L-\hat{x})L - \frac{2}{H-L}\int_L^{\bar{\beta}} (y_O\beta_1-\hat{x})\beta_1 d\beta_1\right]$$

Sufficient conditions for this to be positive are

$$\bar{\beta} \approx L, \quad \text{and} \quad y_O L - \hat{x} > 0$$

The first requires that the lower level of the distribution is close to the upper trigger level, ensuring that the second term in (2.47) is small.

1/ The second order condition requires

$$-3\beta_O^2(H-L) - \delta(\bar{\beta}^3-L^3) < 0$$

and is satisfied for all parameter values since by assumption $\bar{\beta} \geq L$.

The second implies that the first term is positive and may be interpreted by recalling from the definition of the trigger values (2.36)

$$y_0\bar{\beta} - \hat{x} = \sqrt{k}$$

$$y_0\underline{\beta} - \hat{x} = -\sqrt{k}$$

Thus it seems reasonable to argue that values for β greater than the average of $\bar{\beta}$ and $\underline{\beta}$ would yield a positive value for $y_0\beta - \hat{x}$:

$$(y_0 L - \hat{x} > 0) \iff L > (\bar{\beta} - \underline{\beta})/2 \qquad (2.48)$$

The two conditions together mean that values for β at the upper end of the dead band become more likely as risk increases. Given that both of these conditions are fulfilled, y_0, decreases with risk. This in turn implies that both trigger values are increasing with risk, as is the distance between them. The result is in marked contrast to the very risky situation where the trigger values decreased with risk. The explanation is that increases in risk may reduce or increase the probability of reform depending on the position of the cut-off points of the distribution relative to the trigger values.

Clearly, the comparative statics for the slightly risky case are reversed when (2.47) is negative, and sufficient conditions for this to happen may be derived as follows. Since β_1 is drawn from the interval $[L,H]$, it must be no less than L. Therefore, using (2.47), if $(y_0 L - \hat{x}) > 0$,

$$\frac{\partial^2 EU}{\partial y_O \partial_L} \leq \frac{2\delta L}{H-L} \quad (y_O L - \hat{x}') \left[1 - \frac{2(\bar{\beta}-L)}{H-L}\right]$$

A sufficient condition for this to be negative is

$$\bar{\beta} \geq (H + L)/2$$

If this condition is fulfilled, y_O increases with risk, and both trigger values, and their difference falls. This condition, means that an increase in risk increases the chance that a reform will be undertaken. This characteristic is shared by the very risky case.

Summarising the results of the analysis of the slightly risky case, it is apparent that the effect of changes in risk can work in either direction. If the lower bound of the distribution lies in the upper half of the dead band, and is close to the upper trigger level, the optimal value for y falls with risk. However, if the lower bound of the distribution is in the upper half of the dead band, and the upper trigger level is in the upper half of the distribution, the optimal y increases with risk. This reversal arises because in the first situation increasing risk reduces the probability of reform, but in the second it is increased.

The comparative statics of changes in k are quite straightforward. k only influences y_O via its effect on $\bar{\beta}$. From (2.36) the partial effect of an increase in k is to increase $\bar{\beta}$. Inspection of (2.46) reveals that an increase in $\bar{\beta}$ must be associated with a reduction in

y_0 therefore

$$\frac{dy_0}{dk} < 0$$

Returning to (2.36) increasing k and reducing y_0 has an overall upward influence on $\bar{\beta}$ but the effect on $\underline{\beta}$ is indeterminate.

$$\frac{d\bar{\beta}}{dk} > 0 \quad , \quad \frac{d\underline{\beta}}{dk} \gtrless ?$$

and from (2.37) the width of the dead band has increased. Therefore, increasing k shifts the dead band upwards, and increases its width, the overall effect on the lower trigger level being indeterminate.

2.5 Concluding Remarks

The implications of the simple decision theory model discussed in this chapter may be summarised in two categories. Firstly, if data is costly, surveys and reforms take place simultaneously, and the government must decide the date of the next survey at the time of the last reform. The resulting optimal duration will be longer, the greater are the data costs and will probably be shorter if the world is more uncertain. However, the latter result may be reversed if future expected benefits are greatly eroded by risk. Secondly, if the change in the policy instrument is costly, the timing of reforms may be governed by announcing trigger values. As long as the state variable remains within the dead band delimited by these trigger values, old policies are allowed to persist. Comparative statics then reveal the following: if reform costs increase, the dead band expands; if the government's priors shift, the dead band shifts in the same direction; and if uncertainty increases the dead band both shifts and changes in width, the direction of these effects depending on the riskiness of the environment.

The advice which policy makers could draw from the analysis relates to proposals for institutional change. The rigidity which bureaucracies impose on public sector decision-making may have quite beneficial stabilising effects: politicians are prevented from making sudden policy switches, and the consequent adjustment costs are avoided. The question of timing of policy reforms may thus be translated into one of optimal bureacratic inertia. If the economic environment becomes more hostile (σ^2 increases) the analysis suggests that a narrower dead band, or more frequent reforms, would be appropriate; thus it would be quite legitimate to require the institutions of decision making to become more flexible or responsible.

Appendix 2.I

Two propositions are presented to show that the decision strategies assumed in the text are optimal. The first shows that with costly data it is optimal to decide on the date of the next reform at the time of the most recent survey. The second shows that when information arrives freely, it is optimal to use a strategy where reforms are triggered by the value of the state variable.

Proposition 2.A.1

When new information does not arrive between reforms, there can be no improvement on the plan formulated at the date of the last reform.

Proof

At time $t = 0$, the optimal plan for the date of the next reform satisfies

$$\underset{\Theta_0}{E(0)} J(\beta, M^*) \geq \underset{\Theta_0}{E(0)} J(\beta, \tilde{M})$$

where expectations formulated at time 0 are based on information Θ_0, M^* is the optimal reform plan, and $\tilde{M}$ is any other plan.

At some subsequent period t, before the planned reform date $(0 < t < M^*)$, no new information has arisen $(\Theta_t = \Theta_0)$. Then M^* is still the optimal plan at time t

$$\underset{\Theta_0}{E(t)} J(\beta, M^*) \geq \underset{\Theta_0}{E(t)} J(\beta, \tilde{M})$$

$\square$

Proposition 2.A.2

No decision rule can perform better than the optimal trigger strategy where reforms are contingent on the value of the state variable.

Proof

Let the decision at time t be represented as follows

$y_t = 1 \Rightarrow$ reform at time t

$y_t = 0 \rightarrow$ do not reform at time t

The best of all possible decision rules chooses y_t^* at every period such that

$$\underset{\Theta_t}{EJ}(\underset{\sim}{\beta}, y_t^*, \underset{\sim}{y}^*) \geq \underset{\Theta_t}{EJ}(\underset{\sim}{\beta}, \tilde{y}_t, \underset{\sim}{\tilde{y}}^*) \tag{2.A.1}$$

where $\underset{\sim}{y}^*$ represents the vector of subsequent optimal decisions, and $\tilde{y}_t$ and $\underset{\sim}{\tilde{y}}^*$ are any other trajectory for the instrument. The optimal trigger values are implicitly defined by

$$\underset{\beta_t}{EJ}(\underset{\sim}{\beta}, 1, \underset{\sim}{y}^*) = \underset{\beta_t}{EJ}(\underset{\sim}{\beta}, 0, \underset{\sim}{y}^*) \tag{2.A.2}$$

and reform takes place if and only if

$$\underset{\beta_t}{EJ}(\underset{\sim}{\beta}, 1, \underset{\sim}{y}^*) \geq \underset{\beta_t}{EJ}(\underset{\sim}{\beta}, 0, \underset{\sim}{y}^*) \tag{2.A.3}$$

Under the assumption that the state variable follows a random walk, β_t is sufficient to represent all relevant information in Θ_t

$$\Theta_t \iff \beta_t$$

Therefore, the trigger strategy (2.A.3) is equivalent to (2.A.1) and no other rule can perform better.

$\square$

CHAPTER THREE

ACTIVE LEARNING

CHAPTER THREE: SUMMARY

Theoretical models are presented in which the current decision influences the quality of information availible in the future. The decision-maker has objectives defined on a target variable which is influenced linearly by one instrument. Uncertainty results from imperfect knowledge of the parameters determining the influence of the instruments on the target. The decision-maker is allowed to choose not only the means of the instruments, but also the degree of policy variation.

Particular emphasis is placed on the analysis of risky situations, and the 'safe port' is defined as the course of action which minimises the variance of the target variable. Under some circumstances, covariances may be used to offset exogenously determined uncertaities.

In a two period linear quadratic example, with a single instrument, numerical simulations are used to describe the solution. In all of the examples examined, the optimal value for y was no lower than its single-period counterpart. It increased with $\hat{x}$ and T, and decreased with M but the effects of σ^2 and s^2 varied between examples. Policy variation was not found to be desirable.

3.1 Introduction

The acknowledgement of ignorance frequently leads to the prescription of caution. With imperfect information about the parameters of the economic system, risk averse economists often recommend piecemeal or marginal reforms instead of global optimisation. Rather than seek to identify the optimum on the basis of uncertain parameter estimates, a less ambitious problem is addressed, namely the direction of welfare-improving changes. Since Lipsey and Lancaster (1956) a considerable literature has accumulated emphasising that limited movements towards the global optimum may not always be desirable but that some categories of second best reforms may be beneficial. For example, Green (1962), and Bertrand and Vanek (1971), describe the conditions under which reducing the biggest distortion increases welfare, and Dixit (1975) provides a simple unified approach for the derivation of these, and other related results. However, one issue which has received little attention, is that of step length: having identified a desirable direction for reform, how far should you go?

This chapter concentrates on the importance of information for the choice of step length and allows the nature of the information revealed to depend on the policies chosen. Short steps are desirable in that their outcomes are relatively certain; however, long steps are riskier and hence their outcomes provide more new information. Also, emphasis is placed on situations where information is very poor, and the notion of the 'safe port' is introduced to describe the vector of policies which the government would implement in cases where risk tends to infinity.

In the model which follows, time is divided into periods and sub-periods. Policies must be set at the beginning of each period, and are fixed until the beginning of the next period. Information, in the form of observations of the realised value of the target variable accrues every sub-period. For example, the government might be able to change taxes only at a budget, but the national accounts are revised more frequently. The government is allowed to introduce policy variation over a period. This variation takes the form of specifing two values for the instrument each being applied to half the sub-periods within the period. In practice these policy variations might be announcments that a particular instrument will change within the forthcoming period: the rate of VAT on luxuries might be set at 10% to be reduced to 5% in 6 months time. Alternatively, policy variation might be achieved by charging different tax rates to different regions.

Active learning is well known in the control literature. Kendrick (1981b) for example, distinguishes between three kinds of optimal control strategies associated with different attitudes to learning. The least sophisticated is an open-loop control which bases the future trajectory of instruments entirely on the data availible at the starting period; no revisions are made in the light of the new data and nothing is learnt. The second strategy involves passive learning and is called feedback control; parameter estimates are revised on the basis of new data but the deployment of instruments is independent of their expected effect on future parameter estimates. Active learning is the most sophisticated procedure and is an example of "closed-loop" control: policies are deliberately used to perturb the system in order more quickly to learn their influence on the state of the world. Kendrick (1982) shows, by means of simulations in a small US macro model, that active learning will occur when there is enough uncertainty in the model, and when there is a sufficiently high terminal penalty. The analysis here provides some analytic explanation of how this result comes about.

The connection between current actions and future information is not new in economics: it is essential to concepts such as 'learning by doing', Arrow (1962); 'experimental consumption', Kihlstrom, Mirman and Postlewaite (1984) and the 'Rothschild effect'. For example, Rothschild (1974) envisages a consumer faced with a two-armed bandit. He is uncertain about the expected payoffs from playing each arm so his actions are influenced by the inclination to play the arm which he believes to pay out most often, and also by the desire to improve his knowledge of the probabilities of winning. Because of sampling errors, the gambler may reach a long-run equilibrium where he plays only the arm with the lower payoff: the general equilibrium repercussions of this phenomenon are examined by Kihlstrom et al (1984). Tonks (1983) and (1984) adds a budget constraint to the model and shows that this may reverse the comparative statics.

Grossman, Kihlstrom and Mirman (1977) present a closely related model. Faced with a new commodity whose characteristics are uncertain, the consumer experiments in the short run, possibly consuming more of it than he would if he were perfectly informed. My model is simpler because firstly more restrictions are imposed on the objective function, and secondly, in section 3.4 OLS rather than Bayesian learning is assumed. The main departure is that the decision-maker may exploit variations rather than holding the policy fixed. In situations where at least one instrument is fixed, covariances provide some potential for the decision-maker to offset the uncertainty relating to the influence of the the uncertain variables.

Section 3.2 gives a preliminary outline of the main argument. Section 3.3 presents the single-perios model and the dynamic model is presented in Section 3.4. Section 3.5 discusses the results.

3.2 Preliminaries

The model envisages a decision-maker who is uncertain about the parameter β which determines the influence of instruments on objectives. Before setting out the model in general terms, the main issues may be sketched using a univariate regression model illustrated in Figure 3.1. This also forms the basis of the example set out in Section 3.4.

The decision-maker is assumed to have a valuation function defined on deviations of the target variable x from some most preferred value $\hat{x}$, but is not sure exactly what value for the instrument y will minimise these deviations. This uncertainty results from the stochastic error term whose variance is assumed known, and also from errors in the estimation of β. If since the beginning of time y has only taken the value zero, the spread of observations along the vertical axis will

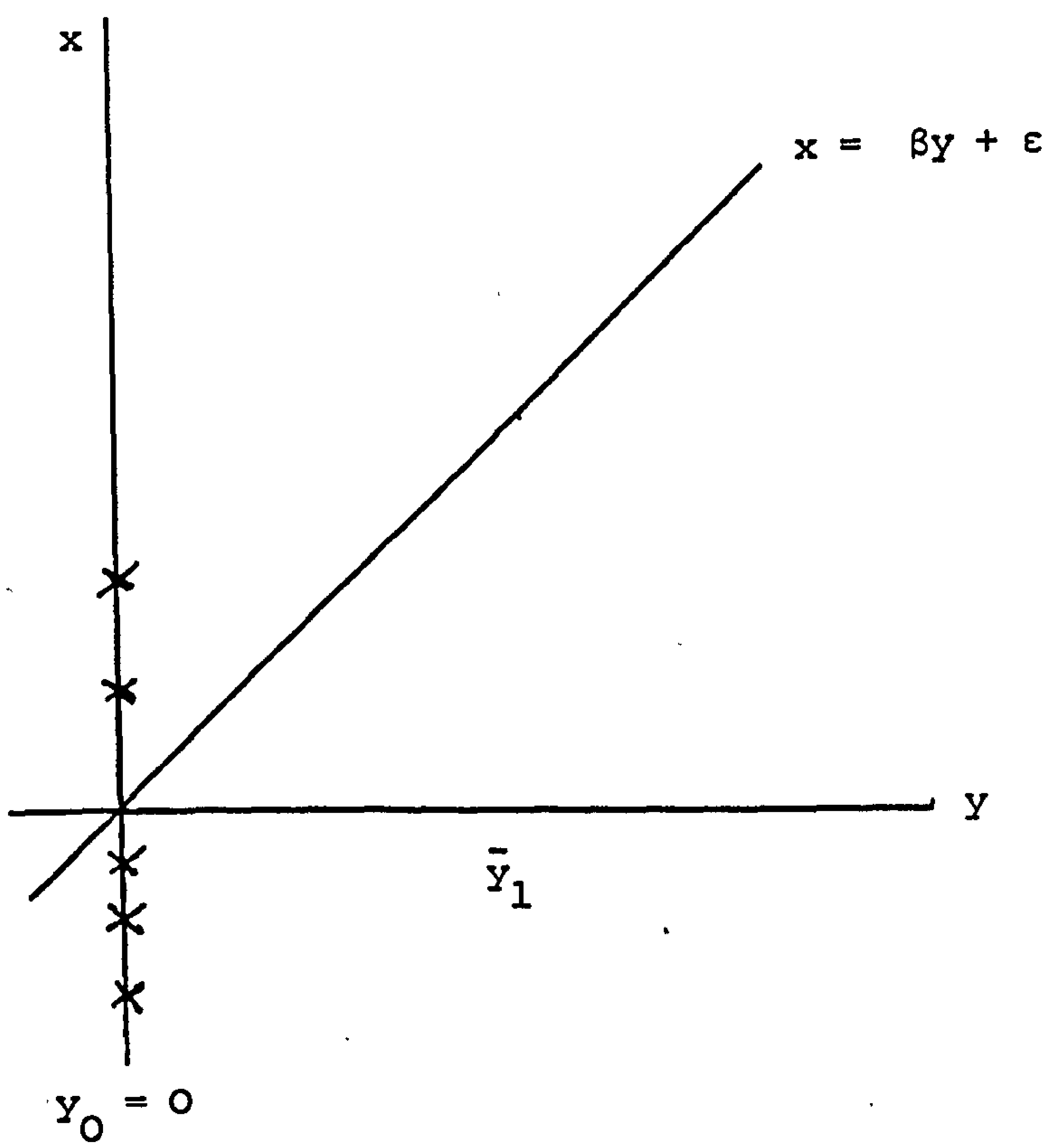


FIGURE 3.1 : Choice of y and variance of β

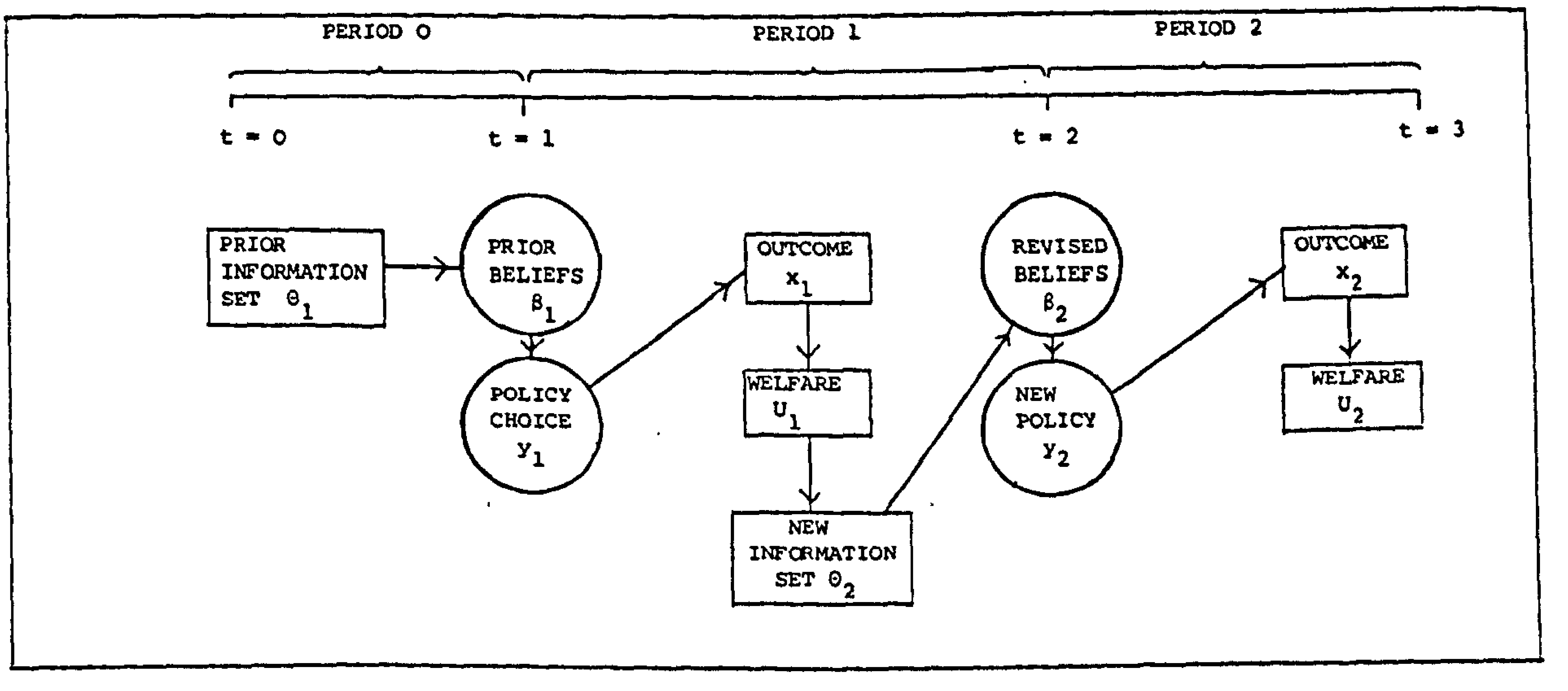
yield some information about the variance of ε, but this is useless information since σ^2 is already known; however it will be uninformative about β, all lines passing through the origin would fit the data equally well. To learn something about β it would be necessary either to allow some variation around y_O, or to fix y at some new level such as $\bar{y}_1$, or employ a combination of both strategies.

In this setting, there will be trade-offs between the use of policies for the purpose of learning, against aiming as closely as possible at the target value of x. More variation in y would lead to increased accuracy of parameter estimates, however it would also entail more variation in x and hence a higher probability in deviating from the target. Deviations from the status quo y_O are rewarded with greater data variation upon which to estimate β, but higher values of y also result in greater variance of x since the uncertainty is multiplicative.

Dynamic issues are obviously essential to the argument. The only reason for concern about the variance of parameter estimates is that they form the basis for future decisions. Choices based on more accurate information have a higher probability of hitting the target, so there will be conditions, relating to the discount rate and structural parameters, when it is beneficial to deliberately aim short of the target this period in order to have a better chance of hitting it next period.

The examination of active learning strategies involving endogenous information requires three time periods. Period zero has the sole purpose of establishing a data set upon which the decision-maker's prior beliefs are formed: no actions are taken until period one. In period two the actions are taken as if there were no learning, since $t = 3$ is the end of time there would be nothing to be gained by obtaining information which could never be used. The essential aspect of learning comes in period one when it is necessary to take account of the influence of current decisions on future information and hence on future policies and outcomes. The order of events is illustratrated in Table 3.1.

TABLE 3.1 : Active Learning: the order of events



3.3 THE SINGLE PERIOD PROBLEM

The model used in this chapter is similar to that used in section 1.4 chapter one. That model is extended to include an intercept, which is subsequently dropped in section 3.4, and also policy variations are introduced. In common with section 1.4, time is divided into three periods. The first establishes a prior data set and contains N sub-periods. The second, period one, lasts for M sub-periods and adds M observations to the prior data set. The final period consists of T-N-M sub-periods. There is no discounting. Decisions must be made at the beginning of a period, and cannot be altered as information accrues from subsequent sub-periods. Not until the beginning of the next period may any new policy be implemented.

As before, the government bases policy decisions on a quadratic objective function defined on deviations from the most preferred value $\hat{x}$ of the target variable x. The objective in period one is therefore to maximise

$$U = -\sum_{t=1}^{M} (x_t - \hat{x})^2 \tag{3.1}$$

In this section an intercept is included in the linear equation describing the structure of the economy

$$x_t = \alpha + \beta y_t + \varepsilon_t \tag{3.2}$$

where $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2$,

and y_t is the value of the government's instrument at time t. It is simple to show that the inclusion of the intercept has a considerable influence on the optimal policy in the single period model, and this provides some indication of the difficulties involved in generalising the models of sections 1.4, and 3.4. Also, the formulae derived form the basis of the decision rules examined by numerical simulation in chapter four.

As well as the intercept, this section introduces a new instrument for the government, namely policy variation. Whereas in the previous exposition of this model (section 1.4), the government had to fix y at a single value over the whole period, here the government is allowed to choose two values for the instrument in the knowledge that each of these will apply to half of the sub-periods over any given period. (Clearly, this requires an even number of sub-periods). The greater is the difference between the two chosen values for y, the greater is the degree of variation being introduced. This formulation is consistent with the interpretation of applying different policies to different groups within the economy. For example, one kind of policy variation observed in developing countries is charging different tax rates to different regions. If the sub-periods of the model are reinterpreted as geographical regions of the economy rather than subdivisions of time, the model is consistent with this interpretation. The important aspect for the following analysis is that the government knows in advance the proportion of the sub-periods (or geographical regions) which will be covered by each distinct policy. This formulation is clearly not equivalent to the assumption that at any sub-period the value of y is a drawing from a distribution whose parameters are chosen by the government. Such an assumption would have the attractive interpretation that the government's choices may be influenced by unpredictable factors: for example the government may be obliged to set instruments in nominal terms even though their real values are subsequently affected by unpredictable changes in the economic environment. This alternative assumption proves to be analytically intractable because of the randomness of the instrument, therefore it is examined by simulation methods in chapter four. The analysis in this chapter proceeds with the assumption that the values of the instrument are known in advance (ie the y's are not random).

The single period problem is to choose the two values for the instrument y_1, and y_2 to maximise the expected value of (3.1). Notice that the expectation of (3.1) may be decomposed in the following way

$$E\sum(x_t-\hat{x})^2 = E\sum(x_t-\bar{x})^2 + M(\bar{x}-\hat{x})^2 \qquad (3.3)$$

The cross-product term involving $E\sum(x_t-\bar{x})(\bar{x}-\hat{x})$ is zero because $\bar{x}$ is the expectation of x. Thus the objective function is made up of the variance of the target variable and squared deviations of its mean from the most preferred value. If in addition $\hat{x}$ is set equal to $\bar{x}$, the government's objective is simply to minimise the variance of the target variable, ie to choose the policy whose outcome is the least uncertain. This policy is termed the safe port.

Using (3.2), the objective function may be written

$$E\sum_{t=1}^{M}(x_t-\hat{x})^2 = E\sum_{t=1}^{M/2}(\alpha+y_1\beta+\varepsilon_t-\hat{x})^2 + E\sum_{t=M/2+1}^{M}(\alpha+y_2\beta+\varepsilon_t-\hat{x})^2 \tag{3.4}$$

Notice that the error term may be taken out of the brackets because its product with the other terms is expected to be zero.

$$E\sum(x_t-\hat{x})^2 = \frac{M}{2}E(\alpha+y_1\beta-\hat{x})^2 + \frac{M}{2}E(\alpha+y_2\beta-x)^2 + M\sigma^2 \tag{3.5}$$

Defining $x_i = \bar{\alpha} + \bar{\beta}y_i$ this may be written

$$E\sum(x_t-\hat{x})^2 = \frac{M}{2}E\sum_{t=1}^{2}[\alpha-\bar{\alpha}+y_i(\beta-\bar{\beta})+\bar{x}_i-\hat{x}]^2 + M\sigma^2 \tag{3.6}$$

Expanding and taking expectations of each term

$$E\sum(x_t-\hat{x})^2 = Mv(\alpha) + \frac{M}{2}(y_1^2+y_2^2)v(\beta) + \frac{M}{2}\sum(\bar{x}_i-\hat{x})^2 + M(y_1+y_2)v(\alpha,\beta) + M\sigma^2 \tag{3.7}$$

where

$$v(\alpha) = E(\alpha-\bar{\alpha})^2$$
$$v(\beta) = E(\beta-\bar{\beta})^2$$
$$v(\alpha,\beta) = E(\alpha-\bar{\alpha})(\beta-\bar{\beta}) \tag{3.8}$$

The optimal policy in the single period model minimises this expression. Having written the objective function in terms of parameter estimates, variances, and instruments, it is straightforward to describe optimal policy as a function of beliefs about the structural parameters of the economy. This description is summarised in the following propositions.

<u>Proposition 1</u>

<u>The introduction of policy variation is never desirable in the single period model, so the optimum always has</u> $y_1 = y_2$

Proposition 1 is obvious from the concavity of the objective function. A proof by contradiction might proceed as follows: assume that $y_1 \neq y_2$ is optimal. Concavity implies that the payoff from the average of the y's is greater than the average of the payoffs from each y. Therefore setting both y's equal to $(y_1+y_2)/2$ would be preferred to the original assumption. Thus $y_1 \neq y_2$ cannot be optimal.

<u>Proposition 2</u>

<u>The optimal choice of</u> y <u>is in general</u>

$$y_1 = y_2 = -\frac{v(\alpha,\beta) + (\bar{\alpha}-\hat{x})\bar{\beta}}{v(\beta) + \bar{\beta}^2}$$

Proof

Using proposition 1 (3.7) may be written

$$E\sum(x_t-\hat{x})^2 = Mv(\alpha) + My_1^2v(\beta) + M(\bar{\alpha}+\bar{\beta}y_1-\hat{x})^2 + 2My_1v(\alpha,\beta) + M\sigma^2 \qquad (3.9)$$

The first order condition for a minimum is

$$\frac{\partial}{\partial y_1} E\sum(x_t-\hat{x})^2 = 2My_1v(\beta) + 2M(\bar{\alpha}+\bar{\beta}y_1-\hat{x})\bar{\beta} + 2Mv(\alpha,\beta) = 0 \tag{3.10}$$

which is satisfied by the formula in the proposition. The second order condition is satisfied since

$$\frac{\partial^2}{\partial y_1^2} E\sum(x_t-\hat{x})^2 = 2Mv(\beta) + 2M\bar{\beta}^2 > 0 \tag{3.11}$$

Notice that the optimal choice of y does not depend on either σ^2 or $v(\beta)$. Theil's (1957) certainty equivalence theorem applies to these sources of uncertainty. In the linear quadratic model changing y does not affect these types of additive uncertainty, therefore they are irrelevant to the decision. However, it is possible to exploit the covariance between α and β. This argument may be made clear by considering the choice of y whose outcome would be least uncertain.

<u>Proposition 2a</u>

<u>The safe port for</u> y <u>is</u>

$$y_1 = y_2 = -\frac{v(\alpha,\beta)}{v(\beta)}$$

The proof simply sets $\hat{x} = \bar{x}$ in (3.7) and proceeds as for proposition 2. And given zero covariance, proposition 2b is obvious.

Proposition 2b

With zero covariance the safe port is

$y_1 = y_2 = 0$

This is a special case of proposition 2a. The safe port is solely concerned with reducing the variance of x, and does not attempt to control its mean. Therefore, with no covariance, uncertainty about β is completely avoided by setting y = 0. However, when there is non-zero covariance, as in proposition 2a, it is possible to offset some of the uncertainty which relates to the imperfect estimate of β . Consider an example where it is known that α's higher that expected are associated with β's higher than expected (there is positive covariance). If y were set to zero, uncertainty about the target variable would depend on the variance of the error term, and the variance of the estimate (having assumed that the estiamte of the intercept is independent of the errors). Given positive covariance the choice of a small negative value for y would add an amount to x when α turned out to be lower than expected, and subtract an amount when α was higher. This would reduce the variance of the target compared with the choice of y = 0. Proposition 2a gives the optimal value for y to exploit the covariance.

More generally, the government would have preferences about the mean of x as well as its variance. The model used in chapter 1 section 1.4 assumes in addition that the intercept is known to be zero, and thus there is no covariance. This special case forms the basis for section 3.4, and the optimal choice of y in the single period model may be derived by making the appropriate restrictions to proposition 2.

<u>Proposition 2c</u>

<u>With no intercept and no covariance the optimal choice of y is</u>

$$y_1 = y_2 = \frac{\hat{x}\bar{\beta}}{v(\beta) + \bar{\beta}^2}$$

This formula is familiar from chapter one. With $v(\beta) = 0$, the government aims at the centre of the target, and as $v(\beta)$ increases they aim lower.

3.4 THE DYNAMIC PROBLEM

With the restriction that the intercept of the structual equation is zero, the model set out in the previous section is very similar to that used in chapter one section 1.4. The difference is that in period one the policy-maker now chooses the two values for the instrument rather than the duration of the period before reform. This makes no difference to the single period problem faced in the final period because the government's best policy is to set the variance of the instrument to zero. Therefore the choice of y in the final peiod, and hence the maximised value of the objective function is identical in this model to that of section 1.4.

Maintaining the notation defined in chapter one, the minimised value of final-period losses is given by chapter one's equation (1.35). The objective in period one is therefore to minimise

$$J = E_1 \sum_{t=N+1}^{N+M} (x_t - \hat{x})^2 + (T-N-M)[\hat{x}^2 E_1 \frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2} + \sigma^{-2}]$$

(3.12)

Substituting, expanding, and taking expectation through the first term,

$$J = M[v(\beta_1)(y_1^2+y_2^2)/2 + \hat{x}^2 + \sigma^2 - \hat{x}\bar{\beta}_1(y_1+y_2)]$$

$$+(T-N-M)[\hat{x}^2 E_1 \frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2} + \sigma^2] \tag{3.13}$$

The first order conditions for optimality of y_1, and y_2 are

$$\frac{\partial J}{\partial y_i} = M[v(\beta_1)y_i - \hat{x}\beta_1] + (T-N-M)\hat{x}^2 E \frac{\partial}{\partial y_i} \frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2} = 0$$

for $i = 1,2$ (3.14)

Interpretation of these conditions obviously depends on the expectation of the derivative of $v(\beta_2)/[v(\beta_2)+\bar{\beta}_2^2]$. If the expected value of the derivative is zero, then all that remains of the first order condition is those terms relating to current benefits, and the period one decision would be made taking no account of the future. Thus the formula in proposition 2c (the single period problem with no intercept) would provide a complete description of the solution. It is fairly obvious that this is the case when some of the parameters take limiting values. For example, if $s^2 \to \infty$ or $\sigma^2 \to 0$, the model is one of perfect information. In this case, the derivative is clearly zero, and the optimal choice 's $y_1 = y_2 = \hat{x}/\bar{\beta}_1$. Similarly, as $\sigma^2 \to \infty$, any learning is impossible so the safe port $y_1 = y_2 = 0$ is optimal. And finally, as M becomes large consideration of the final period becomes less important so the period one policy simply chooses $y_1 = y_2 = \bar{\beta}_1 \hat{x}/[v(\beta_1)+\bar{\beta}_1^2]$. These results are summarised in table 3.2.

As noted in chapter 1 section 1.4, the expectation of $v(\beta_2)/[v(\beta_2)+\bar{\beta}_2^2]$ is difficult to evaluate because of the random variable in the denominator. The problem is compounded in the first order condition because differentiation raises the power of the denominator. Therefore, numerical techniques are used to describe the solution to the model in a series of examples. The formulae used in the simulations are based on the least squares estimates described in chapter 1. From (1.21) and (1.25)

$$v(\beta_1) = \frac{\sigma^2}{s^2} \tag{3.15}$$

$$v(\beta_2) = \frac{\sigma^2}{s^2 + \frac{M}{2}(y_1^2+y_2^2)} \tag{3.16}$$

Table 3.2

Active Learning Model: Results in Limiting Cases

limiting case	$\dfrac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2}$	$y_1 = y_2$
perfect information $s^2 \to \infty$ $\sigma^2 \to 0$	0	$\hat{x}/\bar{\beta}_1$
complete uncertainty $\sigma^2 \to \infty$	1	0
no future M=T-N	?	$\dfrac{\bar{\beta}_1 \hat{x}}{v(\beta_1)+\bar{\beta}_1^2}$

$\bar{\beta}_1$ is set exogenously, and the formula for $\bar{\beta}_2$ is derived from (1.24) as follows. Fixing y_i at y_1 and y_2 for half of the sub-periods each

$$\bar{\beta}_2 = \bar{\beta}_1 + \frac{\frac{M}{2}(y_1^2+y_2^2)\nu + y_1 \sum_{i=N+1}^{N+M/2} \varepsilon_i + y_2 \sum_{i=N+M/2+1}^{N+M} \varepsilon_i}{s^2 + \frac{M}{2}(y_1^2+y_2^2)} \qquad (3.17)$$

The random variables are normalised by letting

$$\nu = \sigma f/s \qquad (3.18)$$

$$\sum_{i=N+1}^{N+M/2} \varepsilon_i = \sigma\sqrt{M}g/\sqrt{2} \qquad (3.19)$$

$$\sum_{i=N+M/2+1}^{N+M} \varepsilon_i = \sigma\sqrt{M}j/\sqrt{2} \qquad (3.20)$$

where f, g, and j are standard normal random variables. Now letting h be another standard normal random variable, and using $af+bg = \sqrt{(a^2+b^2)}h$,

$$\bar{\beta}_2 = \bar{\beta}_1 + \sqrt{\left(\frac{M\sigma^2(y_1^2+y_2^2)}{2s^2[s^2+\frac{M}{2}(y_1^2+y_2^2)]}\right)}\, h \qquad (3.21)$$

These equations mean that the objective function may now be written in terms of exogenous parameters, choice variables, and a single random variable. These are listed in table 3.3.

Table 3.3

Notation

parameters	choice variables	random variable
T	y_1	h
N	y_2	
M		
$\bar{\beta}_1$		
σ^2		
s^2		
$\hat{x}$		

The first question to be addressed is whether the government would ever introduce policy variation in period one: ie would $y_1 \neq y_2$ ever be optimal. The first term in the objective function (3.13) is quadratic. Its convexity would tend to favour no policy variation. If the second term were convex also, the objective function as a whole would be convex and $y_1 = y_2$ would be optimal. Ideally, it would be desirable to examine the convexity of $v(\beta_2)/[v(\beta_2)+\bar{\beta}_2^2]$ with respect to y_1 and y_2 for all combinations of y_1 and y_2. However, to reduce the scale of the problem, attention is restricted to loci where $y_1 = y_2$. This provides the following information:

if $v(\beta_2)/[v(\beta_2)+\beta_2^2]$ is convex at $y_1 = y_2$ then $y_1 = y_2$ will be (at least) a local minimum.

Figures 3.2a to 3.2c display plots of $v(\beta_2)/[v(\beta_2)+\beta_2^2]$ against y and show that the function is usually convex but becomes concave for some parameter values when y is small. In these concave cases, the desirabiity of $y_1 = y_2$ cannot be judged without examining the whole of the objective function.

Tables 3.4 and 3.5 examine two cases in more detail, and provide estimates of the second derivatives of $v(\beta_2)/[v(\beta_2)+\bar{\beta}_2^2]$ and the objective function. The accuracy of these estimates, based on 40 drawings of random numbers, may be assessed with the aid of t statistics which are the ratio of the estimate to its computed standard error. The use of antithetics (pairing the results of iterations based on h and its antithetic -h)

Figure 3.2a

$\bar{\beta}_1=1,\ \sigma^2=1,\ M=5$

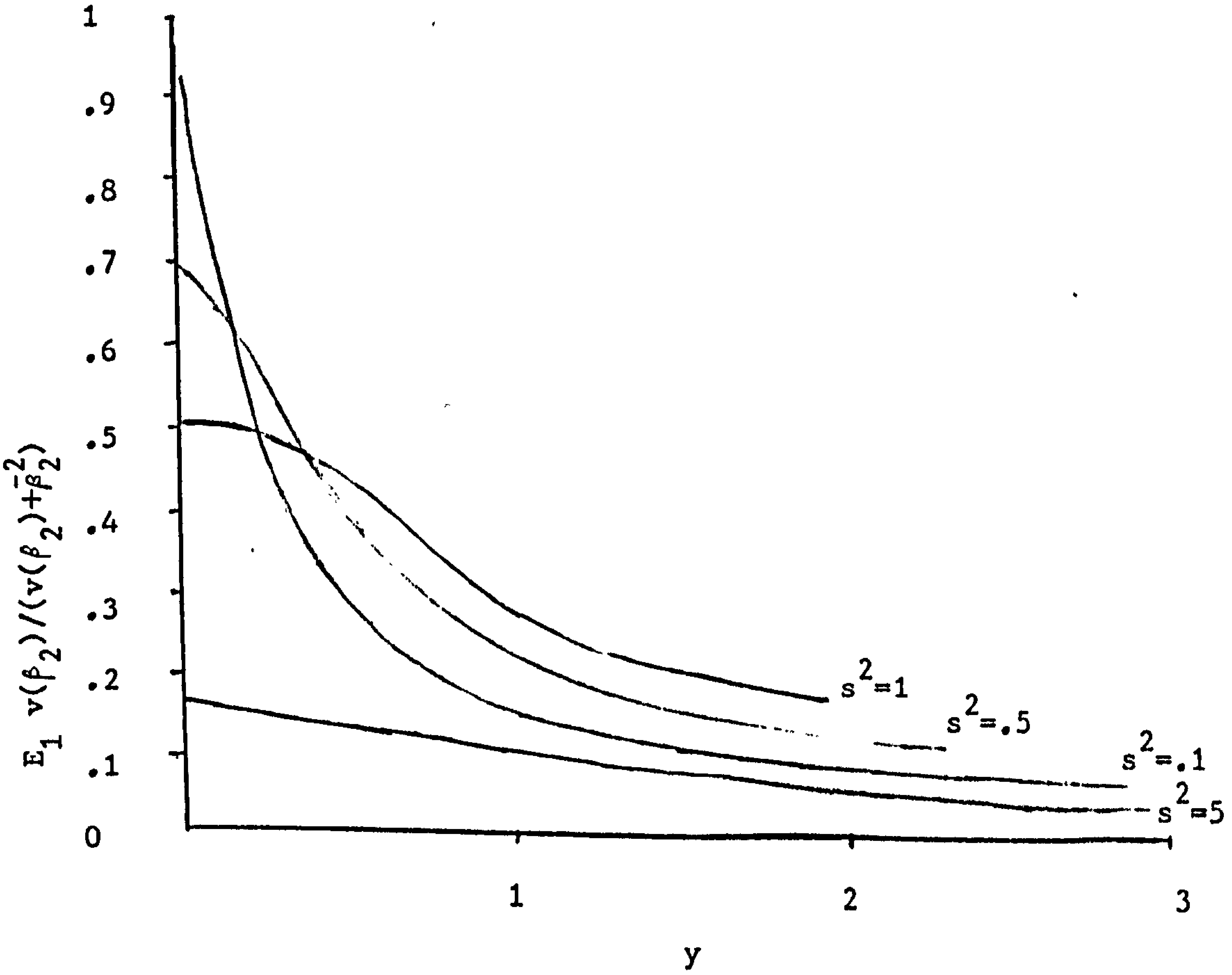
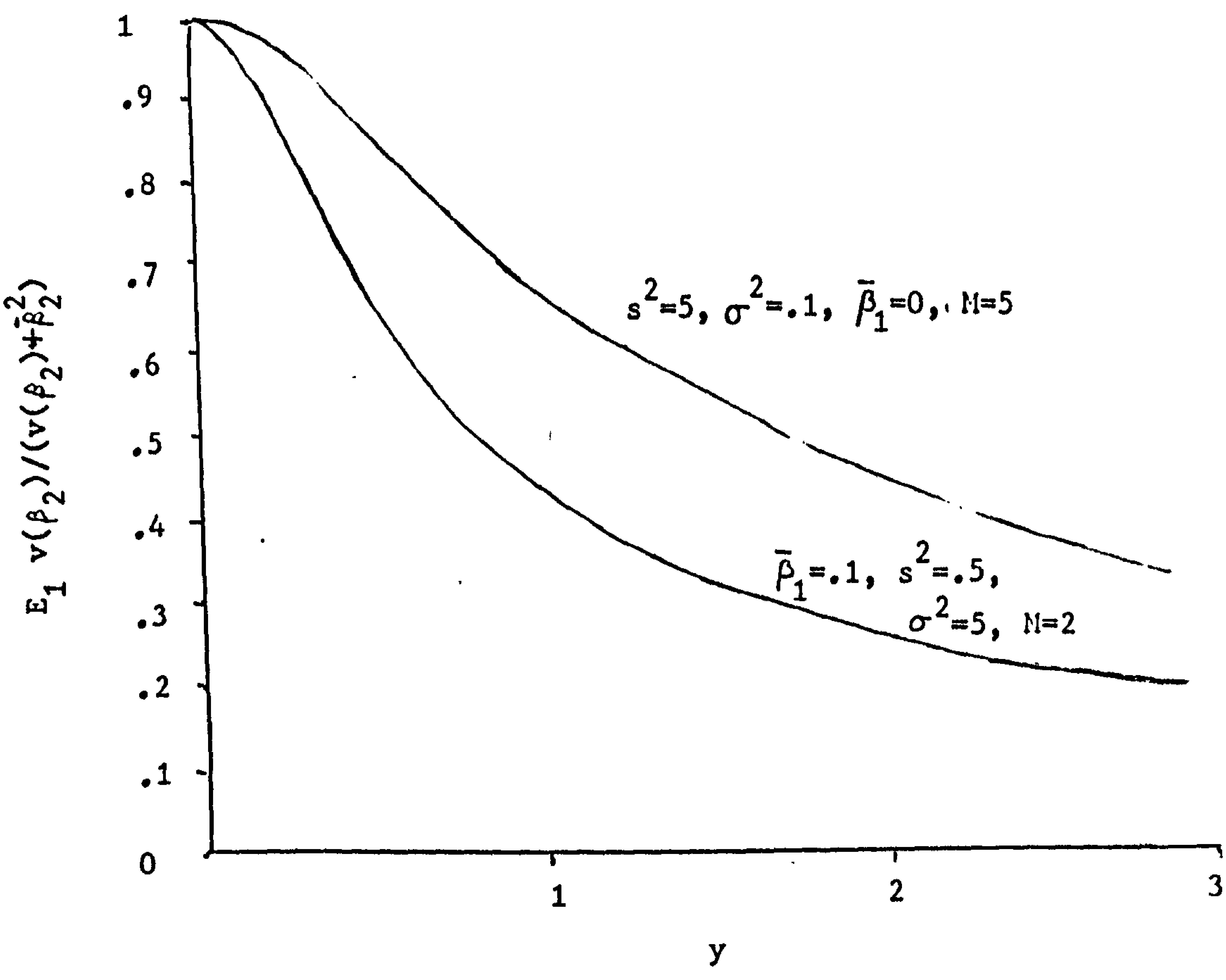


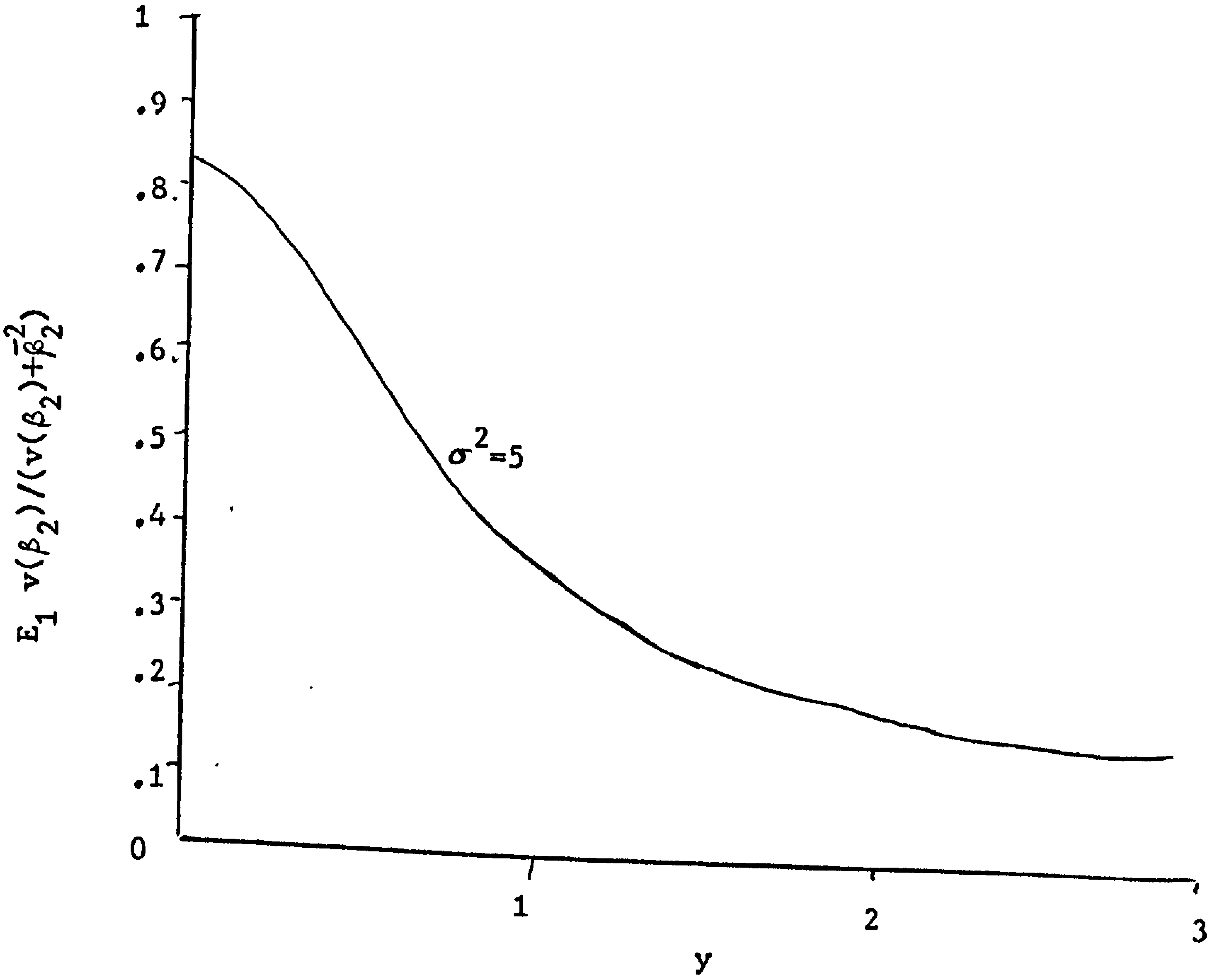
Figure 3.2b



1
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
1
2
3
y
$E_1\ v(\beta_2)/(v(\beta_2)+\bar{\beta}_2^2)$
$s^2=5,\ \sigma^2=.1,\ \bar{\beta}_1=0,\ M=5$
$\bar{\beta}_1=.1,\ s^2=.5,$
$\sigma^2=5,\ M=2$

Figure 3.2c

$\bar{\beta}_1=1,\ s^2=1,\ M=5$



was extremely useful in reducing the standard errors. Table 3.4 shows that when $s^2 = \sigma^2 = 1$ the second derivative of $v(\beta_2)/[v(\beta_2)+\bar{\beta}_2^2]$ is negative and significant up to $y = 0.6$ (excluding $y = 0$). After this point its second derivative changes sign, and the estimate is significant for values of y greater than 1.2. The objective function may also have a negative second derivative for small values of y. In this example $\partial^2 J/\partial y^2$ is negative and significant when $y = 0.2$ or 0.4. This seems to occur when y is a long way away from the optimum and consequently the quadratic section of the objective function is very steep and has little curvature. On the basis of this evidence it seems unlikely that $y_1 \neq y_2$ could be a desirable policy.

The first and second derivatives of the objective function were programmed, and solutions for y_1, and y_2 were obtained by Newton-Raphson (the estimates of the solution are updated at each iteration by subtracting the product of the vector of first derivatives with the inverse of the matrix of second derivatives). At each iteration the expected values of the derivatives were obtained by averaging over the derivatives based on 40 drawings from a standard normal distribution (20 being antithetics). The algorithem converged rapidly, in no more than twelve iterations, and was halted when the squared update for each y was less than 10^{-4}. Despite considerable experimentation with starting values and parameterisations of the model, I was unable to produce

Table 3.4

Active Learning: Concavity of the objective

1	2	3	4
0	.50	33.09 (1.27)	.29 (.88)
.2	.49	-17.41 (-2.34)	-.34 (-3.68)
.4	.47	-23.60 (-3.12)	-.42 (-4.44)
.6	.40	2.47 (1.15)	-.09 (-3.50)
.8	.33	10.21 (6.45)	0 (0.13)
1.0	.28	11.49 (10.79)	.01 (1.40)
1.2	.24	11.44 (16.92)	.02 (2.12)
1.4	.21	11.18 (26.09)	.01 (2.76)
1.6	.19	10.95 (39.83)	.01 (3.45)
1.8	.17	10.76 (60.09)	.01 (4.26)

Parameters: $s^2 = \sigma^2 = 1$, $T = 30$, $N = M = 5$, $\hat{x} = 2$

col 1 $y_1 = y_2$

2 $E_1 v(\beta_2)/(v(\beta_2)+\bar{\beta}_2^2)$

3 $E_1 \partial^2 J/\partial y^2$, (t values in brackets)

4 $E_1 \partial^2 (v(\beta_2)/(v(\beta_2)+\bar{\beta}_2^2))/\partial y^2$, (t values in brackets)

Table 3.5

Active Learning: Concavity of the objective

1	2	3	4
0	.67	-16.19 (-.55)	-.39 (-1.03)
.2	.61	-122.57 (-5.55)	-1.72 (-6.23)
.4	.45	9.41 (1.42)	-.07 (-.92)
.6	.33	20.93 (8.00)	.07 (2.27)
.8	.26	19.76 (17.30)	.06 (4.17)
1.0	.21	18.23 (30.66)	.04 (5.43)
1.2	.18	17.18 (47.66)	.03 (6.04)
1.4	.15	16.49 (66.71)	.02 (6.04)
1.6	.13	16.05 (87.91)	.01 (5.75)
1.8	.12	15.75 (112.29)	.01 (5.36)

Parameters: $s^2 = .5$, $\sigma^2 = 1$, $T = 30$, $N = M = 5$, $\hat{x} = 2$

col 1 $y_1 = y_2$

2 $E_1 v(\beta_2)/(v(\beta_2)+\bar{\beta}_2^2)$

3 $E_1 \partial^2 J/\partial y^2$, (t values in brackets)

4 $E_1 \partial^2 (v(\beta_2)/(v(\beta_2)+\bar{\beta}_2^2))/\partial y^2$, (t values in brackets)

an example where the solution was $y_1 \neq y_2$.

The results, consisting of a single optimal value for y, are presented for different parameterisations of the model in tables 3.6 to 3.9.

Table 3.6 experiments with different combinations of s^2 and σ^2. Reducing σ^2 decreases the uncertainty in the model, and reducing s^2 decreases the accuracy of the prior data set. The figures in the table have a general tendency to fall with σ^2 and increase with s^2. This would tend to support the findings of the single-period model that the decision-maker aims lower in riskier situations. However, this conclusion need not always be valid. There are regions where y increases with σ^2 (at s^2 = 2.5, y increases with when σ^2 increases from 0.9 to 1.3), or falls with s^2 (at σ^2 = 1.3, y falls when s^2 increases from 2.5 to 2.9). Table 3.7 takes a closer look at such a region.

Table 3.8 examines combinations of T and σ^2. When T = 10, there is no final period, and the optimal values for y are simply $\hat{x}\bar{\beta}_1/[v(\beta_1)+\bar{\beta}_1^2]$. For a given σ^2, increasing T increases y. This may be explained as follows: T only appears in the last term of the first order condition (3.14). As long as final-period losses are reduced as y increases, a higher y would be chosen than in the single period problem. Increasing T simply scales up this effect by increasing the importance of the

Table 3.6

Active Learning: Optimal values of y

σ^2 \ δ^2	.1	.5	.9	1.3	1.7	2.1	2.5	2.9	3.3
.1	1.30	1.85	1.88	1.92	1.94	1.96	1.97	1.98	1.98
.5	0.69	1.44	1.88	1.84	1.79	1.96	1.98	1.98	1.98
.9	0.52	1.07	1.49	1.83	2.02	1.97	1.74	1.72	1.85
1.3	0.45	1.02	1.22	1.53	1.79	1.97	2.06	1.98	1.75
1.7	0.40	0.96	1.11	1.31	1.55	1.77	1.93	2.03	2.06
2	0.38	0.91	1.07	1.21	1.41				
3	0.32	0.77	0.99	1.09	1.19				
4	0.29	0.69	0.91	1.03	1.11				
5	0.27	0.63	0.83	0.97	1.06				
6	0.25	0.58	0.78	0.91	1.01				

Parameters: $\hat{x} = \bar{\beta}_1 = 1$, $T = 30$, $N = M = 5$

Table 3.7

Active Learning: Optimal values of y

		s^2				
		2.3	2.4	2.5	2.6	2.7
	.9	1.81	1.74	1.74	1.63	1.68
	1	1.97	1.91	1.83	1.76	1.71
2	1.1	2.05	2.02	1.98	1.92	1.85
	1.2	2.05	2.06	2.06	2.02	1.98
	1.3	2.03	2.05	2.06	2.06	2.05

Parameters $\hat{x} = \bar{\beta}_1 = 1$, $T = 30$, $N = M = 5$

Table 3.8

Active Learning: Optimal values of y

		T					
		10	15	35	55	75	95
	.1	1.82	1.83	1.89	1.95	1.99	2.04
	.5	1.33	1.52	1.96	2.21	2.40	2.55
σ^2	.9	1.05	1.22	1.60	1.84	2.03	2.18
	1.3	0.87	1.00	1.29	1.50	1.68	1.84
	1.7	0.74	0.87	1.15	1.32	1.46	1.57

Parameters: $\bar{\beta}_1 = s^2 = 1$, $N = M = 5$

final period. This intuition is confirmed by noting from (3.14) that

$$\frac{\partial}{\partial y} \frac{v(\beta_2)}{v(\beta_2)+\bar{\beta}_2^2} < 0 \Rightarrow \frac{dy}{dT} > 0$$

by the implicit function theorem. Table 3.8 also indicates some more situations where the optimal value of y increases with σ^2. This happens when σ^2 is quite small and T large.

Table 3.9 runs the model for different combinations of M and $\hat{x}$. When M = T-N, the model collapses to a single period so the final column is simply $\bar{\beta}_1\hat{x}/[\bar{\beta}_1^2+v(\beta_1)]$. In these examples, the optimal value of y always decreases with M and increases with $\hat{x}$. The reason for the latter finding is obvious from the first order condition (3.14). If the derivative in the final term is negative, $\hat{x}$ always has a negative effect on the first order condition. However, the effect of M is more complicated since it affects both $\bar{\beta}_2$ and $v(\beta_2)$.

Several extensions to the model are considered in chapter four. First, the addition of an intercept to the structural equation might be expected to have a considerable effect on the results, and might alone produce examples where policy variation is desirable. Second, modelling the instruments as random variables whose distributions the government controls extends the model to situations where the government controls the instruments only with some error. Third, in multi-period

Table 3.9

Active learning: Optimal values of y

		M					
		2	4	6	8	10	25
	2	2.10	1.63	1.42	1.30	1.23	1.00
	3	2.92	2.28	2.01	1.86	1.76	1.50
$\hat{x}$	4	3.67	2.89	2.57	2.39	2.28	2.00
	5	4.37	3.47	3.11	2.92	2.80	2.50
	6	5.04	4.05	3.66	3.45	3.32	3.00

Parameters $\bar{\beta}_1 = \sigma^2 = s^2 = 1$, $N = 5$, $T = 30$

models interactions between subsequent periods might have a significant influence on current decisions.

3.5 Concluding Remarks

The active learning strategy described in the preceeding models may be related to the underlying motivation of the tax reform literature. Atkinson and Stiglitz (1980), Chapter 12, Ahmad and Stern (1984), and Deaton (forthcoming) discuss the idea that policy recommendations may be made on the basis of directions for marginal improvement rather than global optimisation. This approach is desirable in that it is informationally much less demanding, and the recommendations are likely to be more robust with respect to parameter uncertainty, and to the value judgements of the underlying social welfare function. However, some regard tax reform as essentially a strategy of passive learning.

Deaton (forthcoming) expresses this view quite explicitly: inadequacies of the data mean that local approximations of parameters are all that can be obtained. Global optimisation under these circumstances is little more than a leap in the dark. Therefore the limitations of the data provide a strong justification for an incrementalist approach to policy changes: marginal reforms are made on the basis of local approximations, and then as information becomes available about the new locality further reforms can be proposed. This is clearly a passive learning strategy, new information is taken into account but the choice of instruments is independent of their expected effect on future knowledge. The contrasting view, suggested by the analysis in this chapter, is that active learning tax reform strategies, and destabilisation, may be desirable in order to generate more useful data, and hence better informed decisions in the future. Since "nature may not have been kind enough to perform the crucial experiments on our behalf" (Deaton, forthcoming), it may be necessary to take matters into our own hands.

The properties of the optimal reform strategies derived in this chapter may be summarised as follows: in simulations policy variation was found to be undesirable for the two-period quadratic model. However, learning effects usually increased the level of the instrument above its single period counterpart.

CHAPTER FOUR

LEAST SQUARES ESTIMATES AND ACTIVE LEARNING DECISION RULES

CHAPTER FOUR : Summary

Monte Carlo simulations are used to examine the dynamic interactions between OLS parameter estimates and policy rules. The relative performance, according to a quadratic valuation function, of five decision rules is examined. Each rule yields a different time path for the instrument and as a result provides a different data set upon which subsequent estimates are based.

Anderson and Taylor (1976a) is used as a benchmark. They find no evidence to suggest that OLS under a certainty equivalent policy rule produces consistent parameter estimates. Active learning rules which randomise in the early stage of the simulation, perform better than the certainty equivalent rule because they succeed in making faster initial reductions in parameter variances. However, the optimal active learning rule gives up randomisation before the parameters are perfectly identified and is thus less likely than the certainty equivalent rule to attain consistent estimates.

4.1 INTRODUCTION

When the decision-maker is simultaneously learning and making decisions, dynamic issues arise which cannot be incorporated in a simple two (or three) period model. The active learning strategies examined in Chapter Three have quite complicated dynamic interactions with the methods of estimation, or learning, in multi-period models. In order to examine the consequences of these interactions, this chapter compares the performance of several different decision rules by means of numerical simulation.

Issues of choice under uncertainty and learning are central themes in the control theory literature. Theil (1957), and Simon (1956) are frequently cited as examinations of the simplest case, certaintly equivalence, where uncertainty and hence learning have no effect on the optimal decision. However, when uncertainty relates to the parameters of an economic model, the certainty equivalence theorem no longer applies, although in a wide range of circumstances it may form the basis of an adequate approximation. Taylor (1974) finds that both a certainty equivalent rule, and a variant which includes prior information, converge to the optimal rule with probability one. In general, it would be preferable to allow the decision rule to take account of uncertainty. Prescott (1971) for example, describes an "adaptive" decision rule (hereafter referred to as risk averse) which "would be optimal at time t if experimentation were not a relevant consideration" (p.369). In short, in simulations Prescott finds that this decision rule is preferred to a linear (or certainty equivalent) rule, the latter performing adequately only when uncertainty is very slight. Chow (1973) provides an analytic solution to the optimal feedback control problem under quadratic welfare costs, when the parameters of the econometric model employed are uncertain. A particular case of these risk averse rules was derived in Chapter Three by allowing the variances to approach infinity. Then the policy-maker employs a rule to minimise the variance of the outcome.

Both Chow (1973), and Prescott (1971) refer to the more general problem of simultaneous learning and decision. However, analytic solutions to this class of problems have not been forthcoming, and instead most of the literature has concentrated on providing solution algorithms, such as Chow (1975), MacRae (1975), and Norman (1981)'s modification to the Chow algorithm. These are reviewed in Kendrick (1981b), and some are detailed in Kendrick (1981a). Tse (1975) illustrates his adaptive dual control algorithm with simulations on a third order time invariant linear model with six unknown parameters. The performance of the algorithm is compared with a certainty equivalent rule, and the optimal control based on perfect information. Bar-Shalom and Tse (1976) undertake a similar comparative study of different decision rules including a closed-loop rule. But both of these studies are restricted to quite short time horizons (20 periods) and are therefore not informative about the asymptotic properties of the control and the parameter estimates. An alternative approach is taken by Prescott (1972). Instead of using an algorithm, he suggests a first order moving horizon method as an approximation to the general dynamic programming solution. This method requires at each time period the solution to a two-period dynamic programming problem which is computationally much easier than the n-period ahead problem. In simulations over 6 periods on a linear model with a single unknown parameter Prescott finds that this approximation works better than a myopic (risk averse) rule which in turn is preferred to a certainty equivalent rule.

However, this literature provides little insight into the dynamic interactions between learning and decisions. Anderson and Taylor (1976a) explicitly address the question of how the decision rule affects parameter estimates, however, their paper only examines the certainty equivalent rule. They find that under this rule, the evidence does not suggest that least squares estimates are consistent according to the convergence criterion set out in Anderson and Taylor (1976b). The simulations presented in this chapter are an extension to Anderson and Taylor (1976a). Certainty equivalent (CE), risk averse (RA), variance minimising (VM), and active learning (AL) decision rules are run in a version of their model. The least squares assumption is maintained throughout this chapter.

Some attention has been directed at alternative estimation techniques. Westlund and Stenlund (1982), for example, compare the relative merits of different estimators. In their simulations, based on a model where some regressors are exogenous, and others are instruments, there are situations where an estimator which minimises the sum of absolute deviations, rather than the sum of squares, improves the performance of the certainty equivalent decision rule. This happens because such an estimator places less weight on outliers. In general it would be desirable to allow a more sophisticated estimation technique capable of incorporating information about the process generating the right-hand side variables. However, the least squares assumption makes the problem tractable, and its simplicity may be useful to clarify the issues. Consider th example of a univariate regression model with a constant term. The true model illustrated in Figure 4.1 is assumed to be simply a 45° line through the origin. It is quite possible that the first two observations, which also depend on the random errors, suggest quite different parameter values. If d_1, and d_2, are the first two data points, OLS would fit a line, with slope minus one, through the origin. If, in addition, the policy-maker wants to achieve a target value of zero for x, he sets y equal to zero. Then subsequent observations would be distributed along the x-axis, and there may be no reason for the policy maker to change his initial mistaken belief that the slope of the line is negative rather than positive. Therefore, the interactions between the policy-maker's objectives, and the estimation technique employed, constrain the ability to learn the true parameters of the economy.

In the case of least squares estimation, consistency requires some variation in y. A necessary condition for consistency (according to Theil, 1971) is that the variance of the parameter estimates tends to zero. Once variation in y has been suppressed, this condition cannot be fulfilled. The proof is obvious from Chapter Three, equation (3.30): var y_1 is set to zero, then as M tends to infinity, var β tends to a non-zero quantity.

Inconsistency does not necessarily mean that there is something wrong with the decision rule. However, it raises the question of the relative performance of such a rule against others which give rise to better

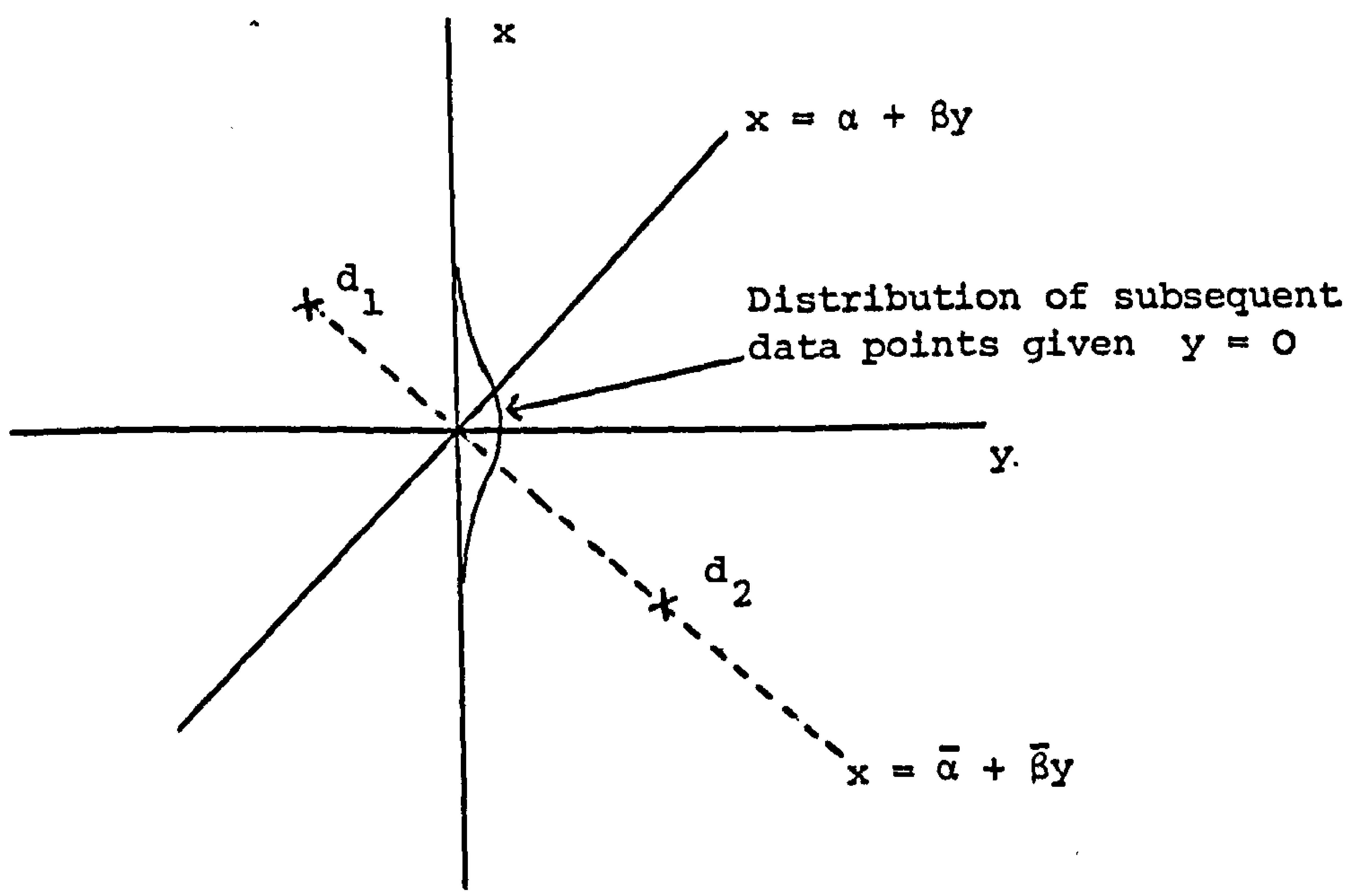


FIGURE 4.1 : The decision based on initial data points leads to false beliefs about the slope.

parameter estimates. The issues addressed by this chapter may be divided into three categories:

i) the performance of active learning decision rules relative to other strategies

ii) the time path of decisions under different rules, given that they yield different data sets

iii) the statistical properties of parameter estimates under different decision rules.

Section 4.2 describes the model and specifies the decision rules; Section 4.3 discusses the choice of simulation technique; the results are presented in Section 4.4 and Section 4.5 adds some concluding remarks.

4.2 THE MODEL

The simulations are based on the univariate linear regression model

$$x_t = \alpha + \beta y_t + \varepsilon_t \qquad (4.1)$$

where x is interpreted as the target variable, and y as the instrument. The parameters are estimated by OLS. For the simulations this is easily done by updating the estimate of the last iteration according to the recursive rules (see Brown et al 1975)

$$\Theta_t = \begin{bmatrix} \bar{\alpha} \\ \\ \bar{\beta} \end{bmatrix}_t = \begin{bmatrix} \bar{\alpha} \\ \\ \bar{\beta} \end{bmatrix}_{t-1} + (Y'Y)^{-1}_t \begin{bmatrix} 1 \\ \\ y_t \end{bmatrix} (x_t - \bar{\alpha}_{t-1} - \bar{\beta}_{t-1}\, y_t) \quad (4.2)$$

where

$$(Y'Y)^{-1}_t = (Y'Y)^{-1}_{t-1} - \frac{(Y'Y)^{-1}_{t-1} \begin{bmatrix} 1 \\ \\ y_t \end{bmatrix} \left[1, y_t\right] (Y'Y)^{-1}_{t-1}}{1 + \left[1, y_t\right] (Y'Y)^{-1}_{t-1} \begin{bmatrix} 1 \\ \\ y_t \end{bmatrix}} \quad (4.2')$$

Similarly, the covariance matrix for the parameter estimates is

$$\text{Cov } \Theta_t = \begin{bmatrix} \text{var } \alpha, \text{cov}(\alpha, \beta) \\ \\ \text{cov}(\alpha, \beta), \text{var } \beta \end{bmatrix} = (Y'Y)^{-1}_t \frac{SSR_t}{t-2} \quad (4.3)$$

where $SSR_t = SSR_{t-1} + (y_t - \bar{\alpha}_{t-1} - \bar{\beta}_{t-1} y_t)^2 / (1 + \left[1, y_t\right] (Y'Y)^{-1}_{t-1} \begin{bmatrix} 1 \\ \\ y_t \end{bmatrix})$.

These rules give the current estimate as a function of the previous estimate, and the new information.

To start the process it is necessary to supply two values for y. The prior estimates of the parameters are then obtained by inverting the resulting two by two matrix. The pair of x's which result from the initial values of y depend on the values which the random error term take. ε is assumed to be normally distributed with mean zero and variance of one.

Three parameterisations of the model are examined. They all share some common elements: the discount rate is 2.5%, the starting value for the sum of squared residuals is 1, and the true model has $\alpha = 0$, and $\beta = 1$. The remaining parameters, y_1, y_2 and $\hat{x}$ are set out in Table 4.1, and the starting values for the covariance matrix are derived from the other assumptions (by inversion of the (Y'Y) matrix). In model 1, the starting values for $\bar{y}$ are very close together and therefore the prior estimates of the parameters are ill-defined. By contrast, model 2 starts with more accurate priors. Model 3 is the same as model 2 but the policy makers target is set to zero; this is Anderson and Taylor's model III, and as will become apparent, the choice of $\hat{x}$ is not a question of normalisation but significantly affects the results.

The valuation function is quadratic in deviations from the target $\hat{x}$, and is discounted by a factor δ over time

$$U = - \sum_{t=1}^{T} \delta^t (x_t - \hat{x})^2 \tag{4.4}$$

Clearly, the optimal decision at any point in time maximises this valuation function, conditional on current parameter estimates. This optimal decision will evolve over time in response to improvements in information, and random errors. As approximations to this optimal decision path, five simple decision rules are examined.

i) Certainty Equivalent (CE). Anderson and Taylor (1976a) discuss this rule, where the policy-maker aims at the target treating

TABLE 4.1

PARAMETERISATIONS OF THE MODEL

	y_1	y_2	$\hat{x}$	var (α)	var (β)	cov (α,β)
M1	1.01	0.9	2	223	249	-236
M2	0	1.0	2	1	2	-1
M3	0	1.0	0	1	2	-1

the current parameter estimates as if they were known with certainty

$$y(CE)_t = \frac{\hat{x} - \bar{\alpha}_t}{\bar{\beta}_t} \tag{4.5}$$

This formula may be derived by setting (4.4) to zero, and using the expectation of (4.1).

ii) Risk Averse (RA). A generalisation of the certainty equivalent rule is to allow the policy-maker to take account of uncertainty. This rule maximises the expected value of the valuation function at time t, and the following formula is obtained by differentiation of (4.4)

$$y(RA)_t = - \frac{\operatorname{cov}(\alpha,\beta)_t - (\hat{x}-\bar{\alpha}_t)\bar{\beta}_t}{\operatorname{var}\,\beta_t + \bar{\beta}_t^2} \tag{4.6}$$

Note that this is a passive learning strategy. Although it takes account of changes in variances from period to period, it ignores the expected influence of current decisions on future data.

iii) A particular case of the risk averse rule was examined in Chapter Three. This arises when the variances become very large. This Variance Minimising (VM) rule minimises the variance of x regardless of the expected deviations from the target.

$$y(VM)_t = - \frac{\operatorname{cov}(\alpha,\beta)_t}{\operatorname{var}\,\beta_t} \tag{4.7}$$

This formula may be derived by minimising the variance of x, as defined in (4.1).

iv) Active Learning (AL). This strategy allows the policy maker to randomise in order to increase the data variations, and hence the accuracy of parameter estimates in the future

$$y(AL)_t = -\frac{\text{cov}(\alpha,\beta)_t-(\hat{x}-\bar{\alpha}_t)\bar{\beta}_t}{\text{var }\beta_t+\bar{\beta}_t^2} + e\left[\phi_1\text{var }\alpha_t+\phi_2\text{ var}\beta_t+\phi_3\text{cov}(\alpha,\beta)_t\right] \quad (4.8)$$

The degree of randomisation is conditional on the elements of the current covariance matrix, so that more parameter variance means more randomisation. e is assumed normal with mean zero and unit variance. Clearly, this is a generalisation of the risk averse decision rule, which has assigned all of the weights ϕ to zero.

The following procedure was used to select the weights for the active learning decision rule. ϕ_3 was set to zero a priori. ϕ_1, and ϕ_2 were optimised by grid search, using short runs (20 periods, and 5 replications) of model 1. The outcome was

$\phi_1 = 0.001$

$\phi_2 = 0.01$

After 20 periods, the discount factor has fallen to about 0.6, therefore subsequent periods have relatively little effect on the value of the objective function, though they provide useful information about the paths of the instruments and the parameter estimates in the long run.

v) <u>Quadratic Active Learning</u> (QAL). A generalisation of the active learning strategy discussed above includes terms in squared variances, and an intercept.

$$y(\mathrm{QAL})_t = -\frac{\mathrm{cov}\ (\alpha,\beta)_t - (\hat{x} - \bar{\alpha}_t)\bar{\beta}_t}{\mathrm{var}\ \beta_t + \bar{\beta}_t^2} \tag{4.9}$$

$$+ \max \left\{0, e\left[\mu_1 + \mu_2(\phi_1 \mathrm{var}\alpha_t + \phi_2 \mathrm{var}\beta_t) + \mu_3(\phi_1 \mathrm{var}\alpha_t + \phi_2 \mathrm{var}\beta_t)^2\right]\right\}$$

Maintaining ϕ_1 and ϕ_2 at their values determined for AL, μ_1, μ_2 and μ_3 were chosen (by grid search) to maximise expected welfare on the basis of short runs of model 1. This yields

$$\mu_1 = -0.01$$

$$\mu_2 = -1.1$$

$$\mu_3 = -0.01$$

Clearly some inaccuracy is involved in the choice of weights for AL, and QAL, since they were optimised on a restricted model where the particular drawing of random numbers may have a significant effect on the results. Also, optimisation was over a restricted class of weights,

ϕ_3 was set to zero a priori, and ϕ_1 and ϕ_2 were fixed across both QAL and AL. These restrictions were necessary to reduce the computer time involved in the grid search.

To protect the computer program against exceptional drawings of random numbers, for all decision rules the value of the instrument is restricted to the interval $[\hat{x} - 10, \hat{x} + 10]$. This restriction binds very infrequently. Anderson and Taylor (1976a) found that it had to be invoked about six times per run. With the decision rules examined in this chapter, it is irrelevant for RA, and binds most frequently for QAL. In total (across all decision rules) it is invoked about twenty times per run.

4.3 SIMULATION TECHNIQUE

Despite the capabilities of modern computers, it is nevertheless incumbent upon the researcher to show that good use has been made of computer time. The program was developed in Basic on an Olivetti micro-computer. The results were checked against Anderson and Taylor's model III. Then the program was transferred to Fortran 77 on the University of Warwick IBM mainframe. Besides this programming strategy, some effort was devoted to selecting the best simulation technique.

The choice among alternative simulation methods may have a large influence on the precison of the results. The performance of a given decision rule will depend on the realised values of the random numbers. It is possible to envisage sets of random numbers under which some rules perform well, and others badly. For example, rule I may generally perform better than rule II, but it is still possible that an unlucky draw of random numbers may yield a higher payoff 1/ for II than for I for a number of runs. Therefore, comparisons between decision rules are accompanied by a measure of precision, namely the standard error. Clearly, one way of increasing precision is to replicate the simulation a number of times, using fresh random numbers. Two other methods which are commonly used, and easy to implement, are common random numbers and antithetics.

In principle, it is possible to combine these methods to maximise the precision of a particular comparison, for a given amount of computer time. Although the procedure is clearly laid out in Kleijnen (1974), such optimisation was not undertaken because the simulation model is small enough that there was no binding computer-time constraint. For comparability with Anderson and Taylor (1976a), the number of replications was fixed at 100, however, this leaves open the choice between common random numbers, antithetics or a combination of the two.

Consider the joint application of common random numbers and antithetics.

1/ Payoff is defined as the realised value of the objective function.

$$\underset{\sim}{R}_1 \rightarrow U_1^*(I),\ U_1^*(II)$$

$$-\underset{\sim}{R}_1 \rightarrow U_2^*(I),\ U_2^*(II)$$

$$\underset{\sim}{R}_2 \rightarrow U_3^*(I),\ U_3^*(II)$$

$$-\underset{\sim}{R}_2 \rightarrow U_4^*(I),\ U_4^*(II)$$

$$\vdots \qquad \vdots \qquad \vdots$$

The first set of random numbers is used to calculate the payoffs, denoted U^*, under decision rules I and II. Then, at the second replication, the same random numbers, multiplied by minus one are used. The process is repeated with fresh random numbers for replications 3, and 4, and so on. The statistics of interest are the average difference between the payoffs, and the variance of that difference. Common random numbers reduces variance by introducing positive covariance between $U_i(I)$ and $U_i(II)$. Antithetics reduces variance by introducing negative covariance between $U_i(I)$ and $U_{i+1}(I)$, and between $U_i(II)$ and $U_{i+1}(II)$ However, the joint application of both methods may be counter-productive since there may be overriding negative covariances between $U_i(I)$ and $U_{i+1}(II)$. In order to choose the most appropriate method of estimating the difference between the two payoffs, it is only necessary to perform one pilot run using both common random numbers and antithetics, since this provides estimates of all of the relevant covariances.

However, the choice is still not completely clearcut because the objective of this study is to make comparisons between more than two decision rules in several models. After performing three pilot runs (detailed in Appendix 4.I), the following facts emerged: in comparisons between Al and CE, joint application is preferred in model 1, and antithetics in model 2. However, in each of these cases, the use of common random numbers alone produces standard errors which are no higher than 15% above the best alternative. But comparisons between AL and RA are much more accurate if common random numbers are used. In model 2, each of the other alternatives would produce standard errors more than 3½ times larger than the common random numbers method. This is presumably because the formulae for AL and RA contain some common terms, therefore covariance between them would be expected to be the over-riding consideration.

The choice of simulation technique is, therefore, necessarily a compromise. Common random numbers alone was used throughout because this method was in most cases expected to do not much worse than the best method, but in some cases considerably better than the next best.

4.4 RESULTS

The results may be divided into two categories. Firstly, the performance of the decision rules, and the time paths of the instrument are summarised in Tables 4.2-4.5. Secondly, the consequences of a given decision rule for the parameter estimates are outlined in Tables 4.6-4.8. Before proceeding to a discussion of these tables, it is possible to dispose of the variance minimising rule quite briefly.

4.4.1. The Variance Minimising Decision Rule

No results are presented for the variance minimising decision rule. The numerical simulations revealed that this rule has the unusual property that the time path of the instrument is determined entirely by the choice of starting values for y, and is fixed. Thus, its payoff according to the objective function is only a coincidence: had other starting values been chosen the payoff would be much better or worse. The reason for the constancy of y under this rule may be explained by reference to the recursive OLS equations. (4.3) shows that each element in the covariance matric is updated by multiplication of $(Y'Y)^{-1}$ by the same scalar, therefore the decision rule, $-\mathrm{cov}(\alpha,\beta)/\mathrm{var}(\beta)$ is independent of that scalar. Then it only remains to show from (4.2') that the updating of $(Y'Y)^{-1}$ does not affect the relative magnitudes of the off-diagonal and lower right elements.

$$\text{Let} \quad (Y'Y)_t^{-1} = \begin{bmatrix} Y(1) & Y(2) \\ Y(3) & Y(4) \end{bmatrix}_{t-1} \tag{4.10}$$

Then the variance minimising decision rule is

$$y_t = -\left[\frac{Y(2)}{Y(4)}\right]_{t-1} \tag{4.11}$$

substituting into (4.2') yields

$$(Y'Y)_t^{-1} = (Y'Y)_{t-1}^{-1} - \begin{bmatrix} Y(1) & Y(2) \\ Y(3) & Y(4) \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{Y(2)}{Y(4)} \end{bmatrix} \left[1, -\frac{Y(2)}{Y(4)}\right] \begin{bmatrix} Y(1) & Y(2) \\ Y(3) & Y(4) \end{bmatrix} / f_t$$

$$= (Y'Y)_t^{-1} - \begin{Bmatrix} \left[Y(1) - \frac{Y(2)^2}{Y(4)}\right]^2 & 0 \\ 0 & 0 \end{Bmatrix} / f_t \tag{4.12}$$

where

$$f_t = 1 + [1, y_t]\,(Y'Y)_{t-1}^{-1} \begin{bmatrix} 1 \\ y_t \end{bmatrix}$$

(Note that this requires symmetry, $Y(2) = Y(3)$).

Therefore, only the first element of $(Y'Y)^{-1}$ changes and the value of y_t remains constant over time. This result obviously generalises to the case of a vector of instruments, provided that the required inverse exists. Then Y(1), Y(2), Y(3) and Y(4) are interpreted as partitions of the $(Y'Y)^{-1}$ matrix rather than scalars.

4.4.2 The Instrument

Table 4.2 shows the value of the objective function obtained by each of the remaining decision rules in each model. Results for QAL are presented for model 1 only. In all three models the active learning rule gives higher payoffs than either the risk averse or certainty equivalent rules. Its dominance over CE is not surprising for model 1 since the choice of weights for AL would rule out the possibility that it might produce lower expected benefits. However, the weights were not optimised for model 2, and additional runs of the model (not presented here) confirm that simultaneously increasing both ϕ_1 and ϕ_2 by a small factor would improve the performance of the rule. That is, if AL were optimised for model 2 it would require more randomisation. QAL attains the highest payoff in model 1, but the specification of its weights seem to be quite sensitive to the particular model since, in additional runs which are not reported in the table, it performed quite badly in model 2. RA performs badly, coming last in both models 1 and 2, however, it attains the second highest payoff in model 3.

TABLE 4.2

THE PERFORMANCE OF THE DECISION RULES

	AL	RA	CE	QAL
M1	-51.24	-67.55	-56.87	-48.65
M2	-54.38	-58.55	-56.12	
M3	-40.36	-40.41	-48.01	

The statistical significance of these rankings is displayed in Table 4.3. The difference between the payoffs of each rule is calculated at each replication, and the means and standard errors of these differences appear in the table. Those which are significantly different from zero, according to a t test at the 95% confidence level, are indicated with *. This procedure confirms that for model 1, RA is significantly worse than AL, QAL, and CE, and in addition that QAL is significantly better than CE. Similarly, for model 2, the dominance of AL over RA is significant; and for model 3, CE is significantly worse than all contenders.

Part of the explanation of the relative performance of the decision rules is the speed with which they get close to the target. Table 4.4 shows the proportion of the runs for which the outcome was within an arbitrary small distance ($\sqrt{0.166}$) of the target at time t. The outstanding feature of these results is the good performance of the certainty equivalent rule. For model 1 it is beaten only once, by AL and QAL at time 5; for models 2 and 3 it dominates the other rules. Clearly, the merit of the certainty equivalent rule is that it gets quite close very quickly. Comparing the active learning rules with RA by this criterion, the former dominate in models 1 and 2. In model 3 however, both have very similar performances.

Table 4.5 presents the time paths for the instruments themselves, and their variances across the 100 replications. The distinguishing features are as follows. The risk averse rule initially has a very small variance. This means that the decisions are only slightly affected by the particular drawing of random numbers and results from the importance of variances, rather than new observations in the decision rule. The

TABLE 4.3

DIFFERENCE IN PERFORMANCE

		AL	RA	CE	QAL
M1	AL	0.00 0.00	-16.30* (1.99)	-5.62 (2.89)	2.60 (1.88)
	RA		0.00 0.00	10.68* (2.86)	18.90* (1.75)
	CE			0.00 0.00	8.22* (2.87)
	QAL				0.00 0.00
M2	AL	0.00 0.00	-4.17* (1.26)	-1.74 (3.19)	
	RA		0.00 0.00	2.43 (3.43)	
	CE			0.00 0.00	
M3	AL	0.00 0.00	-0.06 (0.14)	-7.65* (2.31)	
	RA		0.00 0.00	-7.60* (2.29)	
	CE			0.00 0.00	

Standard errors in brackets

* indicates significance at the 95% confidence level

TABLE 4.4

ESTIMATES OF THE PROBABILITY THAT THE OUTCOME IS CLOSE TO THE TARGET

	T	AL	RA	CE	QAL
M1	5	0.29	0.00	0.12	0.31
	10	0.48	0.00	0.51	0.46
	25	0.69	0.03	0.89	0.72
	50	0.82	0.20	0.97	0.83
	75	0.87	0.24	0.99	0.88
	100	0.93	0.25	1.00	0.91
M2	5	0.01	0.01	0.49	
	10	0.21	0.22	0.72	
	25	0.68	0.63	0.94	
	50	0.88	0.78	0.98	
	75	0.93	0.85	1.00	
	100	0.94	0.86	1.00	
M3	5	0.53	0.53	0.60	
	10	0.63	0.64	0.80	
	25	0.73	0.73	0.98	
	50	0.74	0.75	0.99	
	75	0.76	0.74	1.00	
	100	0.80	0.76	1.00	

Close is defined as $(\hat{x}-y)^2 < 0.116$

TABLE 4.5

THE TIME PATHS OF THE INSTRUMENT

		-----AL-----		-----RA-----		-----CE-----		-----QAL----	
	t	y	var y	y	var y	y	var y	y	var y
M1	5	1.38	0.36	0.96	0.00	1.45	4.09	1.33	0.29
	10	1.55	0.27	0.96	0.02	1.67	1.76	1.57	0.25
	25	1.71	0.25	1.03	0.07	1.94	0.07	1.74	0.16
	50	1.83	0.16	1.17	0.16	1.99	0.04	1.84	0.10
	75	1.86	0.13	1.22	0.20	1.99	0.02	1.87	0.09
	100	1.93	0.07	1.22	0.20	1.99	0.01	1.90	0.07
M2	5	0.79	0.25	0.80	0.25	1.81	1.92		
	10	1.13	0.34	1.14	0.32	1.91	0.16		
	25	1.64	0.26	1.52	0.33	1.96	0.05		
	50	1.85	0.12	1.71	0.30	1.99	0.03		
	75	1.89	0.12	1.77	0.26	1.99	0.02		
	100	1.92	0.08	1.78	0.24	1.99	0.01		
M3	5	0.39	0.07	0.39	0.07	-0.08	1.87		
	10	0.35	0.17	0.36	0.17	0.02	0.12		
	25	0.26	0.09	0.27	0.09	0.01	0.04		
	50	0.23	0.09	0.23	0.08	0.02	0.03		
	75	0.21	0.09	0.23	0.08	0.01	0.02		
	100	0.20	0.08	0.22	0.08	0.01	0.01		

Var y is the variance across one hundred replications of the value of the instrument at time t.

opposite is true for the certainty equivalent rule. Since it only depends on the parameter estimates, which over the first few periods are greatly affected by the drawings of random errors, var y is initially high but rapidly falls as the parameter estimates settle down. In model 2, AL and RA are very similar, except that RA maintains a fairly high variance of the instrument. The same is true for model 3 where once again the certainty equivalent rule rapidly converges, this time to zero.

The extent of randomisation involved in the active learning decision rule may be calculated by substituting the values in Table 4.7, or the prior variance in Table 4.1, into the rule itself (4.8). This reveals that in period 3 (the first time when a decision is made) the random element of AL has a variance of about 2.7 but by period 5 this has fallen to 0.04. As expected, the active learning rule introduces a great deal of variance for a very short time.

Similarly, QAL has a variance of 2.9 at time 3, and on average falls to zero by time 5. The extent of randomisation as a function of parameter variance is sketched for AL and QAL in Figure 4.2 Clearly, the important difference is that QAL gives up randomisation quite early, and reverts to a risk averse strategy. Notice also, that although QAL is based on a quadratic it only operates on the upward sloping section of the function.

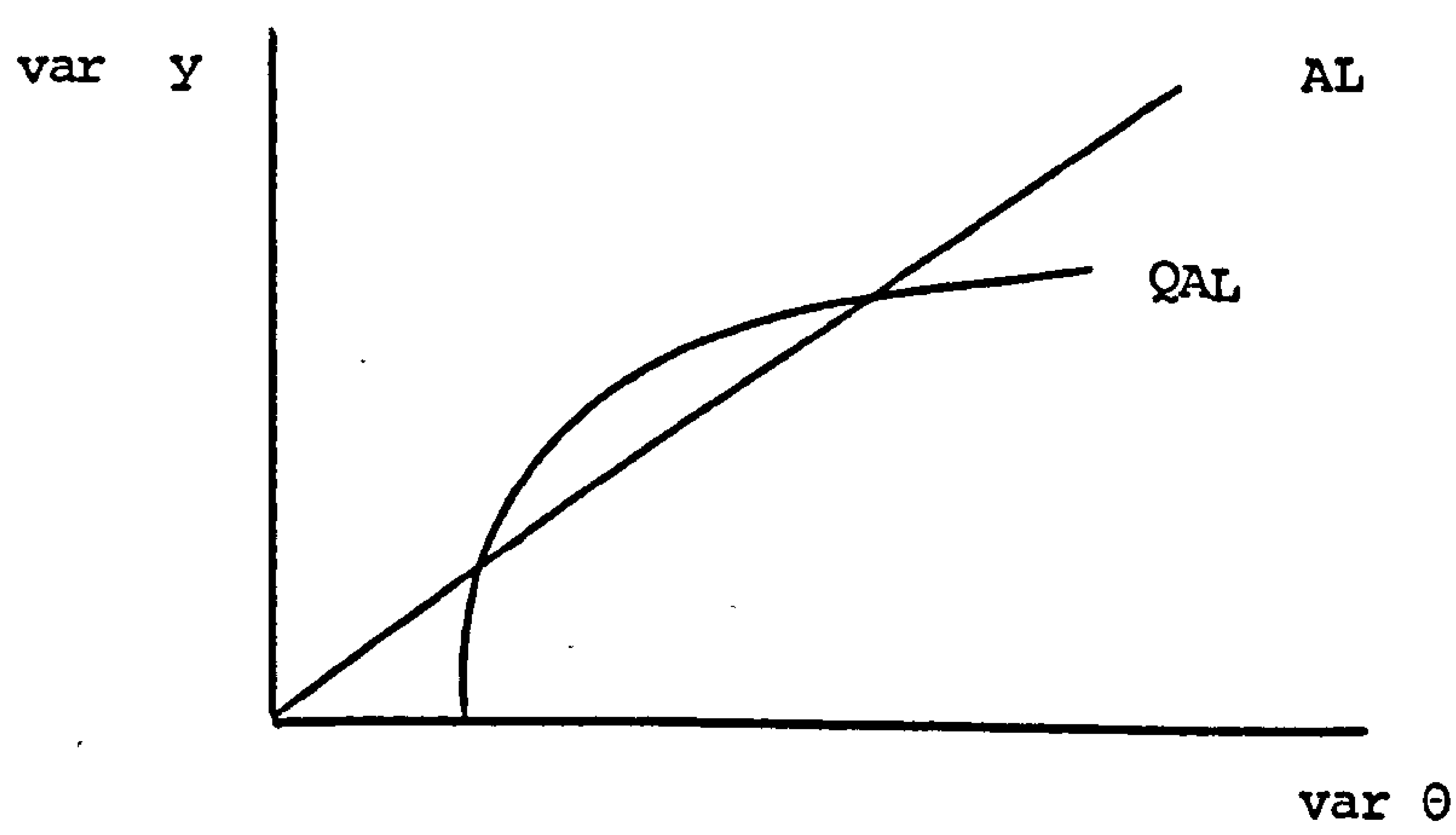


FIGURE 4.2 : Active Learning Decision Rules

4.4.3 Parameter Estimates

Turning now to the properties of the parameter estimates. Table 4.6 shows that the parameter estimates associated with AL, and QAL are generally closer to the true values, than those generated by RA, and CE. In all models CE produces estimates of α with a negative bias, and β with a positive bias. This is consistent with the findings of Anderson and Taylor (1976a, p.1298). The other decision rules do not display such regular biases across all three models. In model 3 however, all of the decision rules produce very good estimates of α. This result is also obtained by Anderson and Taylor (1976b) for the certainty equivalent rule, and may be explained by the normalisation of $\hat{x}$ to zero. Under this assumption all three decision rules converge towards $y = 0$, and thus over time more observations of x are revealed at this point. In terms of Figure 4.1 data points accumulate along the x-axis and therefore the true intercept is identified. Even so, there is no reason to believe that the estimates of β will be consistent, and Anderson and Taylor, on the basis of a longer run for model 3 with the certainty equivalent decision rule argue that it is not. They also state that in models where the assumption that the true value of $\alpha = 0$ is relaxed, the estimates of α would converge as slowly as those of β.

Consistency of the parameter estimates under RA, QAL, and AL remains an open question. There are strong arguments to support the conjecture that under these rules the parameter estimates are less likely to be consistent than under CE. Consistency requires $(Y'Y)^{-1}$ to

TABLE 4.6

PARAMETER ESTIMATES

	t	AL α	AL β	RA α	RA β	CE α	CE β	QAL α	QAL β
M1	5	0.11	0.86	-0.12	0.93	-1.01	1.89	0.16	0.85
	10	0.02	0.91	0.05	0.74	-0.87	1.76	-0.13	1.09
	25	-0.06	0.96	0.32	0.58	-0.44	1.31	-0.17	1.10
	50	-0.13	1.01	0.52	0.41	-0.32	1.19	-0.15	1.05
	75	-0.13	1.02	0.59	0.34	-0.26	1.15	-0.13	1.03
	100	-0.18	1.06	0.58	0.35	-0.27	1.15	-0.15	1.05
M2	5	-0.15	0.98	-0.15	0.99	-0.27	1.31		
	10	-0.18	1.01	-0.18	1.02	-0.30	1.25		
	25	-0.22	1.11	-0.19	1.01	-0.26	1.17		
	50	-0.21	1.10	-0.17	0.99	-0.26	1.15		
	75	-0.20	1.09	-0.15	0.99	-0.25	1.13		
	100	-0.20	1.07	-0.16	1.00	-0.25	1.13		
M3	5	0.05	1.23	0.05	1.24	-0.15	1.67		
	10	0.05	1.35	0.04	1.36	-0.15	1.76		
	25	0.06	1.45	0.07	1.43	-0.09	1.76		
	50	0.05	1.49	0.06	1.46	-0.09	1.74		
	74	0.06	1.51	0.08	1.45	-0.07	1.70		
	100	0.07	1.49	0.11	1.41	-0.05	1.67		

converge to the zero matrix (Anderson and Taylor, 1976a), and the greater the variations in y, the more likely is this condition to be satisfied. Therefore, it might be argued that since CE maintains greater variations in y for a longer period than any of the other rules, it stands a better chance of attaining consistent estimates. However, CE always centres the instrument around the target whereas AL, RA and QAL aim lower than the target according to the extent of uncertainty. Therefore, although the variances of these rules diminish faster than that of CE, the shifts in their means may be sufficient to attain consistency in some cases.

The evolution of the elements of the covariance matrices are displayed in Tables 4.7 and 4.8. Not surprisingly, in model 1, the active learning rules succeed in obtaining the most rapid reductions in parameter variances over approximately the first ten periods. QAL gets the fastest initial reduction, but is rapidly overtaken by AL. Then they are both overtaken by the certainty equivalent rule which finishes with the lowest variances at period 100. This is presumably because the active learning rules use the accuracy of the parameter estimates in the early stages to settle on the best policy. The certainty equivalent rule, which performs worse in the early stages keeps making relatively large changes in y which improve the parameter estimates in the late stages. In all models the risk averse rule has the least well-defined parameter estimates because the inclusion of variance in the decision rule has the effect of damping changes in y. With these small changes in y, the parameter variances remain quite high. In models 1 and 3, RA, AL, and QAL have higher parameter variances in period 100 than CE. Extended runs of model 3 (reported in Appendix 4.II) suggest that by period 3000 CE still has lower parameter variances

TABLE 4.7

PARAMETER VARIANCES

	t	AL α	AL β	RA α	RA β	CE α	CE β	QAL α	QAL β
M1	5	2.36	2.12	143.19	157.27	37.89	41.52	2.20	1.87
	10	1.31	1.00	88.89	97.52	6.69	6.76	1.57	1.20
	25	0.87	0.51	53.55	58.69	0.61	0.27	0.97	0.57
	50	0.60	0.27	41.88	46.08	0.41	0.13	0.70	0.33
	75	0.54	0.21	40.65	44.84	0.35	0.10	0.63	0.27
	100	0.48	0.16	39.75	43.84	0.33	0.09	0.57	0.23
M2	5	1.01	1.80	1.01	1.82	0.77	0.56		
	10	0.67	0.94	0.67	0.96	0.49	0.22		
	25	0.42	0.38	0.44	0.48	0.40	0.13		
	50	0.32	0.16	0.36	0.32	0.35	0.10		
	75	0.28	0.12	0.32	0.28	0.33	0.09		
	100	0.26	0.10	0.30	0.25	0.32	0.08		
M3	5	0.67	2.05	0.68	2.06	0.45	1.01		
	10	0.43	1.54	0.43	1.55	0.19	0.68		
	25	0.26	1.24	0.28	1.28	0.08	0.55		
	50	0.18	0.99	0.20	1.07	0.04	0.45		
	75	0.15	0.88	0.18	1.00	0.03	0.41		
	100	0.13	0.80	0.17	0.96	0.02	0.38		

The parameter variance is the average across one hundred replications of the variance of the parameter estimate at time t.

TABLE 4.8

COVARIANCE (α, β)

	t	AL	RA	CE	QAL
M1	5	-2.03	-149.89	-39.49	-1.84
	10	-1.05	- 93.00	- 6.64	-1.28
	25	-0.61	- 55.99	- 0.38	-0.70
	50	-0.37	- 43.88	- 0.23	-0.46
	75	-0.32	- 42.65	- 0.18	-0.39
	100	-0.27	- 41.70	- 0.17	-0.34
M2	5	-1.10	- 1.11	- 0.50	
	10	-0.66	- 0.67	- 0.28	
	25	-0.34	- 0.38	- 0.21	
	50	-0.21	- 0.28	- 0.18	
	75	-0.17	- 0.25	- 0.17	
	100	-0.15	- 0.23	- 0.16	
M3	5	-0.91	- 0.92	- 0.39	
	10	-0.64	- 0.65	- 0.19	
	25	-0.45	- 0.48	- 0.10	
	50	-0.33	- 0.37	- 0.06	
	75	-0.27	- 0.34	- 0.04	
	100	-0.24	- 0.32	- 0.03	

The covariance is the average across one hundred replications of the covariance between the parameter estimates at time t.

than the other rules, and this is indicative of a stronger tendency for $(Y'Y)^{-1}$ to converge to the zero matrix. However, this is not true for model 2 where AL, and RA attain lower variances of α, but higher variances of β, than does CE. Clearly, this analysis does not provide strong evidence to support the consistency of the estimates in any of the models under any decision rule.

4.5 CONCLUDING REMARKS

The optimal active learning rule has the following properties: firstly, when parameter variances are very large it converges to some non-zero level of randomisation. This is supported by the fact that only the upward sloping secti- of the quadratic approximation was operative,

Secondly, when parameter variances are sufficiently small, the optimal rule stops randomising. This is described by the negative intercept of the quadratic approximation,

and the findings of Kendrick (1982) for a small macro-model.

The results of the simulations may be summarised as follows:

i) there is no evidence to suggest that the certainty equivalent rule generates consistent estimates (Anderson and Taylor, 1976a)

ii) a linear active learning rule performs significantly better than certainty equivalence. However,

iii) the optimal learning rule gives up randomisation before parameter variances have reached zero. At this point, it reverts to a risk averse strategy which, in some cases, would be expected to have less chance of attaining consistent estimates than a certainty equivalent rule.

The implications are quite clear. The optimal decision rule neither requires, nor does it necessarily attain consistent parameter estimates. Therefore, consistency provides very little indication of the likely performance of the estimator in the context of the relevant decision-making problem. The generalisation of these results to estimation techniques, other than least squares, which incorporate the known process generating the right-hand side variables, remains a question for further research.

APPENDIX 4.I

Joint application of common random numbers and antithetics yields the covariance estimates in Table 4.A1 (the underlying formulae are eq. 281-290, Kleinjnen, 1974, pp.226-228). From these, estimates of the variance of the difference between the payoffs of the decision rules are obtained by applying the formulae in Kleinjnen (1974), Table 10, p.221.

TABLE 4.A1

EXPECTED PRECISION OF DIFFERENT SIMULATION METHODS

	M1 I=AL II=CE	M2 I=AL II=CE	M2 I=AL II=RA
COV(U_i(I),U_{i+1}(I))	10.9	-228.8	-228.8
COV(U_i(II),U_{i+1}(II))	-74.2	-362.2	-363.0
COV(U_i(I),U_i(II))	48.0	134.7	415.9
COV(U_i(I),U_{i+1}(II))	46.4	-254.3	-293.7
VAR(U (I))	303.9	370.8	370.8
VAR(U(II))	804.0	1155.6	652.0
VAR(DIFF) - ANTITHETICS	10.7	9.6	16.6
- COMMON RANDOM NUMBERS	10.3	12.8	2.0
- JOINT APPLICATION	8.7	12.0	14.1

APPENDIX 4.II

Extended runs of model 3 (over 3000 periods) show that the elements of the $(Y'Y)^{-1}$ matrix display less tendency to converge to zero under AL and RA than under CE. The time paths of the elements of the matrix are displayed in Table 4.A2.

TABLE 4.A2

THE ELEMENTS OF THE INVERSE OF Y'Y FOR AN EXTENDED RUN OF MODEL 3

	AL		RA		CE	
t	(1,1)	(1,2)	(1,1)	(1,2)	(1,1)	(1,2)
5	0.61	-0.83	0.61	-0.83	0.40	-0.35
10	0.44	-0.68	0.44	-0.68	0.20	-0.20
25	0.28	-0.47	0.28	-0.48	0.09	-0.11
50	0.19	-0.34	0.21	-0.37	0.04	-0.06
75	0.14	-0.27	0.19	-0.33	0.03	-0.04
100	0.12	-0.22	0.17	-0.31	0.02	-0.03
500	0.02	-0.05	0.15	-0.26	0.00	-0.01
1000	0.01	-0.01	0.15	-0.25	0.00	0.00
3000	0.00	0.00	0.15	-0.24	0.00	0.00

	AL	RA	CE
t	(2,2)	(2,2)	(2,2)
5	1.79	1.79	0.87
10	1.55	1.57	0.63
25	1.20	1.22	0.49
50	0.99	1.04	0.39
75	0.84	0.95	0.35
100	0.74	0.90	0.32
500	0.33	0.69	0.21
1000	0.23	0.64	0.18
3000	0.16	0.58	0.14

CHAPTER FIVE

TAXATION AND PARAMETER UNCERTAINTY: THEORY

CHAPTER 5 : SUMMARY

Techniques are described for the incorporation of parameter uncertainty into existing models of optimal taxation. Attention is restricted to cases where only one parameter is imperfectly known. Two questions are addressed:

i) should optimal tax rates based on the assumption of complete certainty be regarded as upper or lower bounds?

ii) Should econometric parameter estimates intended for a particular tax optimisation problem be regarded as upper or lower bounds?

The means of obtaining answers to these questions are summarised in three theorems, the first two being applicable to choices over only a single instrument:

i) the optimal tax rate increases or decreases with the variance of a given parameter according to whether the first order condition is convex or concave in the parameter (Diamond and Stiglitz, 1974).

ii) An econometric estimate should be regarded as an upper or lower bound when the ratio of the second and first derivatives of the first order condition, with respect to the uncertain parameter, is positive or negative.

iii) If each element of the vector of first order conditions is linear in the uncertain parameter, then the vector of optimal decisions is not affected by that uncertainty.

Approximations for the magnitudes of the required adjustments are also provided.

Distinction is drawn between revenue constraints which apply ex ante, or ex post. The former applies at the planning stage, and depends only on expected values, but the latter requires some ex post adjustment, and therefore depends on the uncertain parameter.

5.1 INTRODUCTION

By accident or design economists have almost invariably separated questions of parameter estimation and policy optimisation. As a result, studies of optimal taxation, such as Ramsey (1927), Mirrlees (1971), Stern (1976) and Seade (1977) and the literature on the formulation of empirical models, such as Hendry and Richard (1982) and (1983), have developed quite independently. The former generally treat demand system parameters as if they were certain, or could be known with certainty given information on consumers' characteristics; and the latter discuss criteria for model selection, and the properties of estimators, with little or no reference to their intended use.

Rather than attempting a general unification of parameter estimation and policy optimisation, this chapter concentrates on two connections between them. Given that econometric models provide uncertain estimates, the first question is how to adjust optimal tax rules for uncertainty. A case of particular interest is that of 'certainty equivalence' (Simon, 1956, Theil, 1957) when the policy choice is independent of parameter uncertainty.

The second question is how parameter estimates may be revised in the knowledge that they will be used in a particular policy optimisation problem. The case to be considered is that where the policy rule regards all parameters as if they were known with certainty. Information about uncertainty, and the social welfare function, may then be incorporated in a 'certainty equivalent parameter estimate'. In practice, this might reveal whether, in the context of a specific policy choice, a particular parameter estimate should be regarded as an upper or lower bound.

As the methods for adjusting optimal taxes for uncertainty are described, it will become apparent that general results are hard to find. The outcome will depend on the source of uncertainty, the specification of the objectives, and the kinds of policy constraints incorporated in the model. Therefore, rather than establishing general results, the aim of this chapter is to outline techniques which may be applied to specific tax problems in Chapter 6.

5.2 METHOD

For tractability it is assumed that only one parameter is uncertain. This is necessary in order to be able to infer from an increase in the variance (or spread) of that parameter that the riskiness of the environment has unambiguously increased. Clearly, it would be desirable to consider cases of multi-dimensional uncertainty arising from many parameters, however it would then become difficult to say whether a simultaneous increase in some variances, and reduction in others led to an increase or decrease in overall riskiness. Atkinson and Bourginon (1985) have suggested criteria (stochastic dominance) for making such a judgement. However, attention is restricted to the simple case of uncertainty about one parameter only.

The number of instruments available is also an important consideration. The single instrument case is generally more tractable, however, weaker results also apply in the case of many instruments.

5.2.1 A single instrument

In its most general form the criteria for the choice of tax rate may be described by the social welfare function, W,

$$W = W(t, \underset{\sim}{\Theta}) \tag{5.1}$$

where t is the tax instrument and $\underset{\sim}{\Theta}$ is a vector of parameters describing the influence of the tax rate on social welfare. It is assumed, for the time being, that other instruments are fixed, or are determined by rules independent of t, and hence may be regarded as parameters in the present context.

The basis of the methodology adopted here is comparison between a situation of certainty where the welfare function (5.1) applies, with a situation of uncertainty, where decisions are made on the basis of expected welfare.

$$EW = \int_{-\infty}^{+\infty} W(t,\underset{\sim}{\Theta})f(\Theta_j)d\Theta_j \tag{5.2}$$

where $f(\Theta_j)$ is the probability distribution of parameter j.

Diamond and Stiglitz (1974) provide a theorem relating the sign of the difference between the policies which maximise welfare and expected welfare to the concavity or convexity of the first order condition. An approximation due to Malinvaud (1969) for the magnitude of this difference is also exploited. Points other than the maxima of the functions are examined by considering the tax rates which yield a fixed amount of revenue.

Consider the problem under certainty. The first and second order conditions for a welfare maximum are

$$W_t(t,\bar{\Theta}_j,\underset{\sim}{\Theta}_{-j}) = 0 \; ; \; W_{tt}(t,\bar{\Theta}_j,\underset{\sim}{\Theta}_{-j}) < 0 \tag{5.3}$$

where $\underset{\sim}{\Theta}_{-j}$ is the vector of parameters exluding the j^{th}. Given the parameters $\underset{\sim}{\Theta}_{-j}$ and $\bar{\Theta}_j$, t is the optimal tax rate. By holding t at its optimal value and allowing Θ_j to take values other than $\bar{\Theta}_j$ it is possible to plot the first order condition; Figure 5.1 also assumes that it is concave in Θ_j. Now if Θ_j takes values of Θ_j' and Θ_j'' each with probability $\frac{1}{2}$ (there is a mean preserving spread around $\bar{\Theta}_j$), the expected value of W_t falls below zero. From the second order condition, W_t increases as t falls, therefore a reduction in t is required to restore optimality. Thus, an increase in uncertainty about

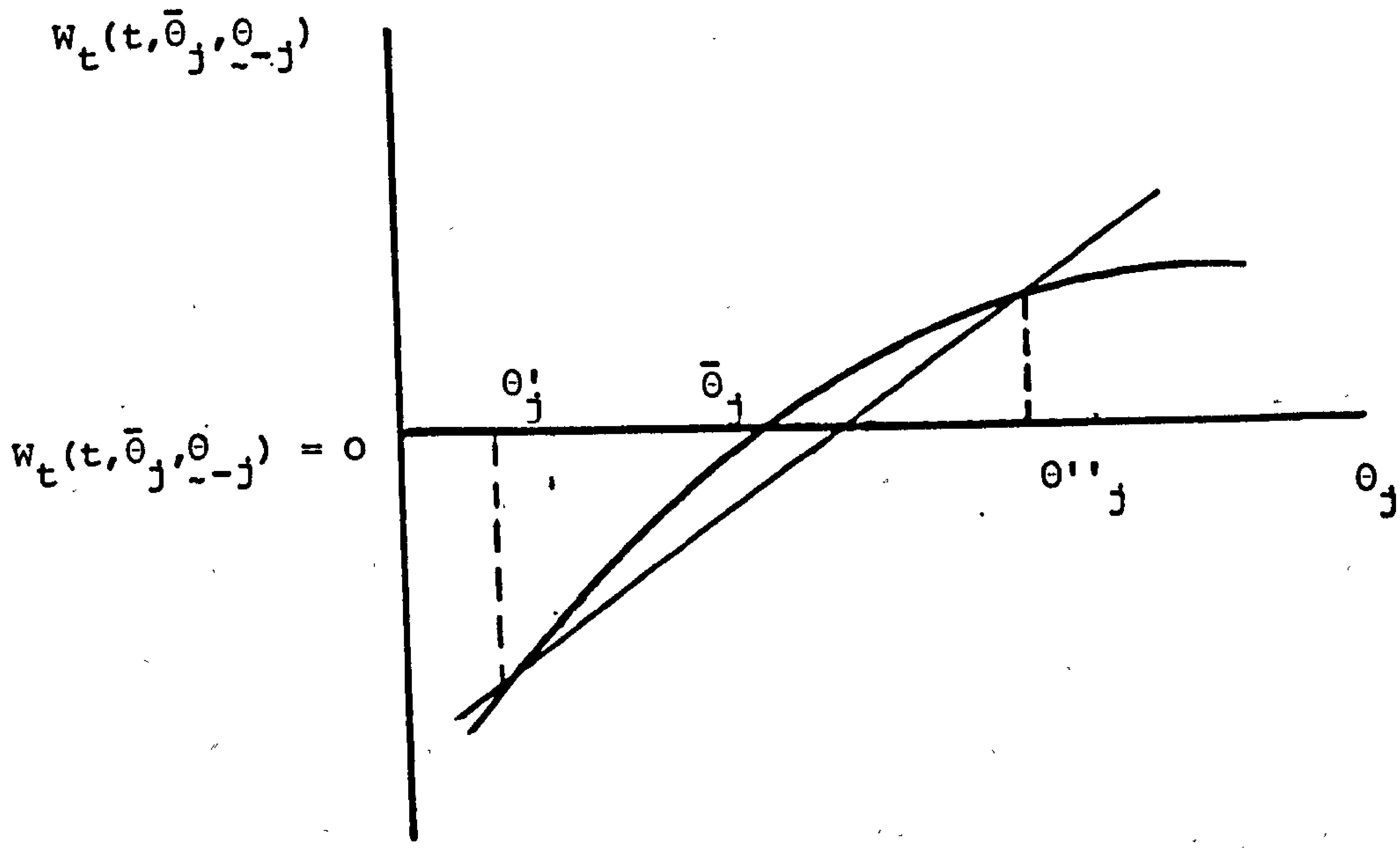


FIGURE 5.1 : Increase in uncertainty about θ_j : the concavity of the first order condition

Θ_j requires an increase (decrease) in t if W_t is concave (convex) in Θ_j. This argument is formalised in Theorem 1.

Theorem 1 (Diamond and Stiglitz, 1974)

The optimal tax rate increases, is constant, or decreases with uncertainty about parameter j according to whether the derivative W_t is convex, linear, or concave in Θ_j, i.e.

$$W_{\Theta j \Theta j t} \gtreqless 0 \Rightarrow \frac{\partial t^*}{\partial \mathrm{var}\Theta_j} \gtreqless 0$$

Note, this is true for more general measures of risk than variance - mean preserving spread for example. Here attention is restricted to means and variances; as indicated by Samuelson (1970), there are a large number of situations where this approach may be regarded as a first approximation to more general classes of distributions. Direct application of Theorem 1 to optimal tax problems reveals when tax recommendations on the basis of certain parameters may be regarded as upper or lower bounds, i.e. it indicates the desired direction of adjustment. The value of the optimal tax rate under uncertainty may be estimated by taking a second order Taylor expansion of the welfare function, around the mean $\bar{\Theta}_j$ of the parameter.

$$EW[t, f(\Theta_j), \underset{\sim}{\Theta}_{-j}] \simeq W(t, \bar{\Theta}_j, \underset{\sim}{\Theta}_{-j}) + \frac{1}{2} W_{\Theta j \Theta j}(t, \bar{\Theta}_j, \underset{\sim}{\Theta}_{-j}) \mathrm{var}\ \Theta_j \qquad (5.4)$$

Differentiate with respect to t, and set both sides equal to zero

$$0 \;=\; EW_t \simeq W_t + \frac{1}{2} W_{\theta_j\theta_j t} \operatorname{var} \theta_j = 0 \tag{5.5}$$

where the arguments of the functions have been suppressed.

The solution to the left-hand side is t^* and to the right is t^{**}.

This may be summarised as follows:

<u>Approximation 1</u> (Malinvaud, 1969)

<u>The solution to the problem under uncertainty may be approximated by adjusting the first order conditions for the problem under certainty in the following way:</u>

$$W_t(t^{**},\bar{\theta}_j,\underset{\sim}{\theta}_{-j}) \;+\; \frac{1}{2} W_{\theta_j\theta_j t}(t^{**},\bar{\theta}_j,\underset{\sim}{\theta}_{-j}) \operatorname{var} \theta_j = 0$$

<u>Then the solution to this adjusted FOC is approximately equal to that of the problem under uncertainty</u>

$$t^{**} \simeq t^*$$

Analogous results apply to the certainty equivalent problem. This requires the introduction of a bias into the estimate of θ_j, such that the resulting certainty equivalent parameter estimate θ_j^* may be used in policy optimisation as if it were known with certainty.

Thus the choice of θ^*_j would ensure that t^* satisfies

$$W_t(t^*, \theta^*_j, \underset{\sim}{\theta}_{-j}) = 0 \qquad (5.6)$$

Theorem 2 shows the conditions under which a parameter estimate should be regarded as an upper or lower bound in the context of a particular optimisation problem.

Theorem 2

The certainty equivalent parameter estimate is above (below) the expectation of the parameter when the ratio of the second and first derivatives of the first order condition is positive (negative).

$$\frac{W_{\theta j \theta j t}}{W_{\theta j t}} \gtreqless 0 \iff \theta^*_j - \bar{\theta}_j \gtreqless 0$$

Proof:

Defining μ as the degree of risk, the first order condition for welfare maximisation is

$$EW_t(t, \underset{\sim}{\theta}, \mu) = 0 \qquad (5.7)$$

Holding $\underset{\sim}{\theta}$ constant and differentiating yields

$$\frac{dt}{d\mu} = - \frac{EW_{t\mu}}{EW_{tt}} \qquad (5.8)$$

By definition, the certainty equivalent parameter estimate is required to satisfy (5.6) which implies

$$\frac{dt}{d\theta_j} = - \frac{W_{t\theta j}}{W_{tt}} \tag{5.9}$$

combining (5.8) and (5.9)

$$\frac{d\theta_j}{d\mu} = \frac{EW_{t\mu}}{EW_{tt}} \frac{W_{tt}}{W_{t\theta j}} \tag{5.10}$$

Since the second derivatives with respect to t are negative for maxima, $d\theta_j/d\mu$ has the same as $EW_{t\mu}/W_{t\theta j}$. The proof of Theorem 1 (Diamond and Stiglitz, 1974) showed by integration that the sign of $EW_{t\mu}$ is the same as that of $W_{\theta j\theta jt}$. Therefore

$$\frac{d\theta_j}{d\mu} \gtreqless 0 \iff \frac{W_{\theta j\theta jt}}{W_{\theta jt}} \gtreqless 0 \tag{5.11}$$

and hence

$$\theta_j^* - \bar{\theta}_j \gtreqless 0 \iff \frac{W_{\theta j\theta jt}}{W_{\theta jt}} \gtreqless 0 \tag{5.12}$$

□

The magnitude of this difference may be estimated using a technique similar to approximation 1.

Expanding (5.6) around $\bar{\Theta}_j$ and equating to zero

$$W_t \simeq W_t(t^*,\bar{\Theta}_j,\underset{\sim}{\Theta}_{-j}) + W_{t\Theta j}(t^*,\bar{\Theta}_j,\underset{\sim}{\Theta}_{-j})(\Theta_j^* - \bar{\Theta}_j) = 0 \qquad (5.13)$$

From Approximation 1,

$$W_t(t^*,\bar{\Theta}_j,\underset{\sim}{\Theta}_{-j}) \simeq W_t(t^{**},\bar{\Theta}_j,\underset{\sim}{\Theta}_{-j}) = -\frac{1}{2} W_{\Theta j\Theta jt}(t^{**},\bar{\Theta}_j,\underset{\sim}{\Theta}_{-j}) \text{ var } \Theta_j \qquad (5.14)$$

Substituting (5.13) into (5.14) yields

Approximation 2

$$\Theta_j^* - \bar{\Theta}_j \simeq \frac{1}{2} \frac{W_{\Theta j\Theta jt}}{W_{\Theta jt}} \text{ var } \Theta_j$$

These theorems and approximations all relate to the maxima of the appropriate welfare functions. A wide variety of tax issues require the examination of other properties of the functions. Consider, for example, the case where the government is required to raise a given amount of revenue. This is illustrated in Figure 5.2 where the revenue under certainty, and expected revenue are plotted for different values of t. The functions are assumed to be concave with unique global maxima at $\hat{t}$ and t^* respectively. In the absence of other constraints it would seem natural to choose the lower of the two tax rates which satisfy the revenue requirement. Therefore, if the expected revenue curve lies everywhere above the revenue curve, uncertainty means a lower tax rate. If, in

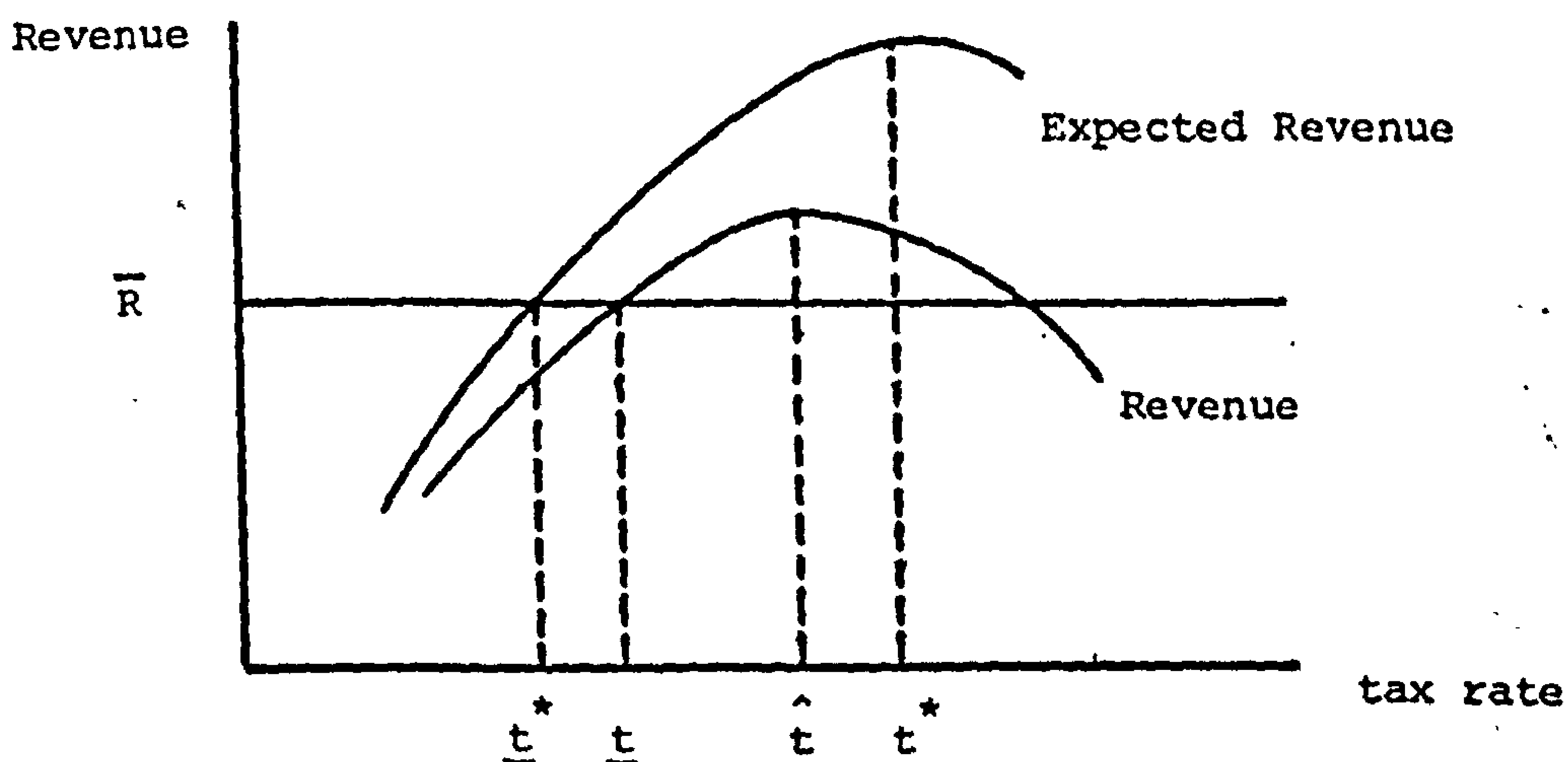


FIGURE 5.2 : Revenue and expected revenue functions

addition expected revenue may be approximated by the Taylor expansion

$$ER(t,\underset{\sim}{\theta}) \simeq R(t,\bar{\theta}_j,\underset{\sim}{\theta}_{-j}) + \frac{1}{2} R_{\theta j\theta j}(t,\bar{\theta}_j,\underset{\sim}{\theta}_{-j}) \text{ var } \theta_j \tag{5.15}$$

then the vertical distance between the revenue and expected revenue curve is given by $\frac{1}{2} R_{\theta j\theta j}$ var θ_j. Hence

Approximation 3

The lowest tax rate $\underline{t}$ which on average raises $\bar{R}$ decreases (increases) with uncertainty if $R_{\theta j\theta j}$ is positive (negative).

$$R_{\theta j\theta j} \gtreqless 0 \Rightarrow \frac{d\underline{t}}{d\text{var}\theta_j} \gtreqless 0$$

5.2.2 Many Instruments

By defining $\underset{\sim}{t}$ as a vector of tax instrument, the Diamond Stiglitz theorem may be generalised as follows. Expected welfare depends on the uncertain parameter θ_j whose distribution $F(\theta_j,\mu)$ has a spread, or riskiness, described by the scalar μ.

$$EW = \int_0^1 W(\underset{\sim}{\theta},\underset{\sim}{t})F_{\theta j}(\theta_j,\mu)d\theta_j \tag{5.16}$$

The first order conditions

$$EW_{ti} = 0 \text{ for all } i \tag{5.17}$$

implicitly define a relationship between the vector of taxes, and the riskiness of θ_j. Defining EW_t as the vector of derivatives of EW, and totally differentiating yields

$$\frac{\partial}{\partial \mu} (EW_t)d\mu + \frac{\partial}{\partial \underset{\sim}{t}} (EW_t)\cdot d\underset{\sim}{t} = \underset{\sim}{0} \qquad (5.18)$$

For a unique global maximum, the matrix of second derivatives of EW with respect to t must be negative definite; therefore (5.18) may be rearranged

$$\frac{d\underset{\sim}{t}}{d\mu} = - (EW_{tt})^{-1} \frac{\partial}{\partial \mu} (EW_t) \qquad (5.19)$$

It is a straightforward generalisation of Diamond and Stiglitz to show that every element in $\frac{\partial}{\partial \mu}(EW_t)$ has the same sign as the corresponding element in $W_{t\theta j\theta j}$ and is zero if $W_{t\theta j\theta j} = 0$. However, this does not necessarily have implications for the sign of $\frac{d\underset{\sim}{t}}{d\mu}$, since there may be over-riding cross effects from the matrix of second derivatives. But, the following theorem is obvious from (5.19)

<u>Theorem 3</u>

<u>If each element of the vector of derivatives of</u> $EW(\underset{\sim}{\theta},\underset{\sim}{t})$ <u>is linear in</u> θ_j <u>then the vector of optimal instruments is not affected by risk.</u>

$$(W_{t\theta j\theta j} = \underset{\sim}{0}) \Rightarrow \left[\frac{\partial}{\partial \mu} (EW_t) = \underset{\sim}{0}\right] \Rightarrow \left(\frac{d\underset{\sim}{t}}{d\mu} = \underset{\sim}{0}\right)$$

Given uncertainty and many instruments, the government's revenue constraint requires careful interpretation. It is impossible to know in advance exactly how much revenue will result from the application of a particular set of tax instruments, therefore the revenue constraint which would be applied under certainty must be modified. There are two alternatives:

(i) Ex ante revenue constraint. The government is required to plan to raise a given amount of revenue R. This may be written

$$\underset{\sim}{t}.\bar{\underset{\sim}{x}} = R \tag{5.20}$$

where $\underset{\sim}{t}$ is the vector of tax rates and $\bar{\underset{\sim}{x}}$ is the vector of expected demands for each commodity.[1/] Because the world is risky and errors are made in forecasting consumers' responses, it is unlikely that the planned revenue target will be precisely fulfilled; however, under this interpretation the constraint states only that the target would be met if the forecasts turned out to be exactly true.[2/]

ii) Ex Post revenue constraint. Under this interpretation exactly the right amount of revenue is always recovered because one instrument t_n is determined ex post. This is written

$$t_n = \frac{R - \sum_{i=1}^{n-1} t_i x_i}{x_n} \tag{5.21}$$

Footnotes for page 158

1/ Note that when the demand functions are linear in parameters it is possible to write the ex ante revenue constraint simply in terms of the expected values of the uncertain partameters.

2/ In models where the government revenue constraint is equivalent to a social resource constraint, it is necessary to make the additional assumption that there exists a resource which may be transfered over time. (Without this assumption a shortfall of government revenue would violate the resource constriant.) The motivation for considering the ex ante cconstraint is that such transfers may be costly. For example it is sometimes argued that unplanned changes in the Public Sector Borrowing Requirement impose a real burden on subsequent economic activity.

where the x's on the right-hand side are the actual demands, not forecasts. Usually indirect tax models are applied to consumer demands, and it is difficult in that context to imagine the tax rate on the n^{th} commodity being determined after the demand for all goods including the n^{th} have been revealed. However, this kind of constraint is a plausible model in other areas of government policy-making. For example, the scheme of prices at which Area Boards buy electricity from the Central Electricity Generating Board, has elements which are determined ex post (see Electricity Council, 1981). The Area Boards formulate expectations about the prices which they are likely to face, and behave accordingly; then at the end of the year the Central Electricity Generating Board makes adjustments to the announced price schedule to ensure that revenue targets are fulfilled.

The distinction between ex ante and ex post revenue constraints is crucial in determining the influence of uncertainty on optimal taxes. The reason is as follows: the Diamond Stiglitz theorem indicates the importance of the curvature of the first order condition with respect to the uncertain parameter. When demands are linear in the uncertain parameter the ex ante revenue constraint depends on nothing but the expected value of this parameter and contributes no curvature to the programme. However, the ex post revenue constraint depends on the entire probability distribution of the unknown parameters and this makes a difference to the convexity of the first order conditions.

The order of differentiation is a convenient way of capturing these contrasting effects. The ex ante revenue constraint amounts to first differentiating the welfare function twice with respect to the uncertain parameter, and then using the the constraint

$$\frac{d}{dt_i}(W_{\theta j \theta j}) = \frac{\partial}{\partial t_i} W_{\theta j \theta j} + \frac{\partial}{\partial t_n} W_{\theta j \theta j} \frac{dt_n}{dt_i} \tag{5.22}$$

Notice that the term dt_n/dt_i, when the revenue constraint enters, is not differentiated with respect to θ. But if the order of differentiation is reversed

$$\frac{\partial^2}{\partial\theta_j\partial\theta_j}(W_{ti}) = \frac{\partial^2}{\partial\theta_j\partial\theta_j}\left(\frac{\partial W}{\partial t_i} + \frac{\partial W}{\partial t_n}\frac{dt_n}{dt_i}\right) \tag{5.23}$$

curvature of dt_n/dt_i is incorporated and hence the revenue constraint has been imposed ex post.

5.3 CONCLUDING REMARKS

In principle it would be possible to avoid any distinction between parameter estimation and policy optimisation. Rather than using statistical criteria to derive estimates from the data, and then selecting optimal taxes according to a welfare function, the whole procedure may be condensed into one operation. Policies may be chosen to maximise welfare given observed data, making policy rules and parameter estimates redundant. This chapter makes only a small step towards such a unified approach however it does draw attention to two pieces of information which may be neglected as a result of the separation of estimation and optimisation.

Firstly, parameter uncertainty may be included in optimal policy rules; and secondly, the 'best estimate' depends on the kinds of policy choice for which it is intended.

The techniques which have been set out in this chapter may be directly applicable to certain categories of policy decisions. In an institutional setting where policies are subject to a large number of constraints it is possible that only one degree of freedom remains. In electricity pricing, for example, although there are a multitude of different rates for different kinds of supply, their relationship to each other is severely constrained. For practical purposes, the standard domestic supply price may be taken as the basis from which other rates are derived. In addition, if there is predominantly one source of uncertainty, such as the price elasticity of electricity demand, the policy problem has the required characteristics and the techniques in Section 5.2 may be applied.

It is difficult to envisage robust generalisations about the influence of uncertainty on optimal taxation. However, before proceeding to examine particular tax models, it is possible to indicate factors which are likely to play an important role in determining the outcome. The Diamond Stiglitz theorem draws attention to the curvature of the first order conditions in the uncertain variable. This in turn is likely to depend on the government's objectives: revenue maximisation would be linear in demands, though not necessarily linear in parameters; and more complex social objectives would be less likely to be linear. Also, the nature of the policy constraints must be taken into consideration: the properties of the solution may be completely different if an ex ante 'planning' constraint is replaced by an ex post 'balancing item' constraint.

CHAPTER SIX

TAXATION AND PARAMETER UNCERTAINTY: EXAMPLES

CHAPTER SIX : SUMMARY

Parameter uncertainty is introduced in six existing models of optimal taxation. The consequences for optimal tax rates, and certainty equivalent parameter estimates are described. The first five models are a direct application of the methods discussed in Chapter Five; the sixth is a more straightforward argument in terms of covariances.

The results, detailed in Table 6.1, are briefly as follows. In the first three models which all examine linear income taxation, the importance of parameter uncertainty depends on the functional form chosen for labour supply. If it is Cobb Douglas, then the optimal tax rate should be adjusted upwards; however if it is linear, only the coefficient on lump sum transfers matters, and its uncertainty can influence the optimal tax rate in either direction. Simulations on the basis of the empirical estimates of Brown et al (1976) suggest that the required adjustment is likely to be small.

Models four to six deal with commodity taxation, four and five being based on the Corlett Hague (1953) model, with a linear expenditure system representing demands. With an ex ante revenue constraint, uncertainty about any of the marginal propensities does not matter, but uncertainty about necessary expenditure on goods one and two reduce and increase the optimal tax on good one respectively. However, when the revenue constraint is applied ex post, uncertainty about any of the parameters is important. Model five shows that with many consumers, and lump-sum transfers, the uniformity result does not hold under uncertainty. And model six, which following Deaton and Stern (1986) allows lump-sum grants, based on household characteristics, shows that uniformity only holds if there is no correlation between errors in observing household characteristics, and social marginal utilities.

6.1 Introduction

Somewhat rough calculations tend to suggest that the empirical significance of parameter uncertainty for real-world taxation decisions is slight. One measure of this importance is the uncertainty surrounding the revenue that would be raised by a small increase in income taxation. Taking Abbott and Ashenfelter's (1976) estimate of the elasticity of labour supply, and using their reported standard error, it is possible to predict upper and lower bounds for the revenue that would be generated by a policy change. For the UK in 1984, these admittedly crude calculations suggest that the revenue arising from an increase of 1p in the pound on the aggregate income tax rate lies (with 95% confidence) between £1.43 bn and £1.44 bn. The difference, which is interpreted as the uncertainty about exchequer revenues, is £7.5 m which amounts to less than 0.1% of PSBR. (Details of the calculations are provided in Appendix 6.I.)

Policy makers who have in the past treated econometric estimates of elasticities as known quantities may draw some comfort from this kind of argument. Although policies are based on imperfect information, the imperfections are not very large, and would probably not have any great influence on the best decision. Therefore, it saves time and trouble if they are ignored. Nevertheless, this chapter proceeds to examine parameter uncertainty in several tax models. In the light of the preceding argument about its likely significance for decision making, such a project requires some justification.

Firstly, Chapter Five suggested that generalisations about taxation and parameter uncertainty may be hard to find. Despite a strong prior belief that the final decision will not be much affected, it is not possible to lay down precise conditions under which the effect will be small. Therefore, each case must be considered on its merits, and every case where uncertainty does not matter may make some marginal contribution to the peace-of-mind of the risk averse decision-maker.

Secondly, the concept of an "ideal type" has guided and contributed to a great deal of economic thought. Indeed, the models upon which this chapter are based may fall into this category. Although uncertainty is a part of real-world decision-making, the exercise of describing what the perfectly rational government would do if they had available all relevant information, is worthwhile. It establishes a convenient vocabulary for discussing broader issues, and also provides a benchmark against which more complicated problems and solutions may be judged. The inclusion of uncertainty in these simple models therefore serves two roles. The more ambitious is to convert simple models into operational devices for giving policy advice. Given likely orders of magnitude, this may not be a very worthwhile exercise. Even so, there remains the more modest aim of providing another benchmark. The ideal type under perfect information already exists, and its usefulness may be enhanced by considering in addition an ideal type with parameter uncertainty.

The tractability of models of optimal taxation relies on their assumption. In the most general Diamond Mirrlees (1971) framework it is difficult to establish any properties of the optimal tax system, and presumably, it would be equally difficult to rule out many of the possible uncertainty adjusted systems. In this unrestricted case the only way to proceed would be by means of numerical simulations (see Stern, 1976, and Heady and Mitra, forthcoming for discussions of this approach in models with perfect information). This chapter takes the alternative approach applying analytic techniques to greatly simplified tax problems. The models may be divided into categories according to the assumptions which they make about the instruments available to the government. The first three models restrict attention to the choice of a single linear income tax rate to maximise revenue; and the remaining three models examine commodity taxation. The assumptions of perfect competition, and constant returns to scale, which are required for Diamond and Mirrlees' "production efficiency" result are implicit throughout. These assumptions ensure that as long as the government's budget constraint is fulfilled, society's resource constraints must be satisfied also.

The six models are described in detail in Section 6.2. The proofs rely on the theorems and approximations set out in Chapter Five. Section 6.3 discusses and summarises the results.

6.2 The Models

6.2.1 Model 1: Cobb Doublas Utility Functions and Diversity of Time Endowments

Broome (1975) proves the following: given Rawlsian objectives with revenue from the linear income tax being equally redistributed across individuals, identical Cobb Douglas preferences, and a lower limit to the distribution of wages, the optimal tax rate is 58.6%. The most obvious way to allow for uncertainty in this model would be to increase the spread of the distribution of wages: optimal taxes depend on the distribution of abilities which is represented with imperfect accuracy by the distribution of wages. However this is difficult to interpret in terms of econometric parameter estimates and their variances, therefore the following method is adopted.

Firstly the model is restated taking differences in time endowments rather than wages, to represent differences in abilities. Secondly, Broome's theorem is shown to hold when preferences take a particular form. And finally, uncertainty about the parameter characterising preferences is introduced and the results compared with Broome's.

If consumers maximise identical utility functions

$$U_h = q_h^{\alpha} \, (T_h - \ell_h)^{(1-\alpha)} \tag{6.1}$$

where $0 < \alpha < 1$

subject to budget constraints

$$pq_h = w\ell_h + m \tag{6.2}$$

where q is consumption of goods, T is time endowments, m is lump-sum transfers, and subscripts are for households, the associated labour supply functions are:

$$\ell_h = T_h - (1-\alpha)m/w \tag{6.3}$$

The government can obtain an econometric estimate of α, which will be required for tax optimisation, by regressing aggregate expenditure on leisure on non-labour income (or alternatively by regressing expenditure on commodities on non-labour income).

Aggregating across households, noting that transfer payments equals average per capita tax revenue, $m = t.\bar{\ell}$, and normalising such that $w = (1-t)$, average labour supply may be written

$$\bar{\ell} = \frac{\bar{T}(1-t)}{1-\alpha t} \tag{6.4}$$

where bars denote averages. Thus the government objective function is

$$R = t\bar{\ell} = \frac{t\bar{T}(1-t)}{1-\alpha t} \tag{6.5}$$

The first order condition[1/] is

$$\frac{\partial R}{\partial t} = \frac{[1+t(\alpha t-2)]\bar{T}}{(1-\alpha t)^2} = 0 \tag{6.6}$$

and yields the optimal tax rate

1/ The second order condition $(\partial^2 R/\partial t^2 < 0) \iff (\alpha < 1)$.

$$t = \frac{1-\sqrt{(1-\alpha)}}{\alpha} \tag{6.7}$$

If $\alpha = 1$ consumers place no value on leisure, and thus their labour supplies are inelastic. Therefore the best tax rate is 100% and the outcome is perfect equality of income. However, for smaller values very high tax rates are counter-productive since the tax base shrinks as individuals decide to consumer more leisure. For $\alpha < 1$, lower tax rates maximise revenue, but the optimal rate never falls below 50%. Note that if information about α is diffuse - it is equally likely that it takes any value between zero and one - its expectation will be 0.5, and if this is substituted into the optimal tax formula a rate of 58.6% emerges. The following propositions examine the sensitivity of this tax rate to uncertainty about α.

Proposition 1.1

Under Cobb Douglas labour supply, the introduction of uncertainty about the expenditure share of goods increases the optimal level of income taxation

$$(0 < \alpha < 1) \Rightarrow \frac{\partial t^*}{\partial \mathrm{var}\alpha} > 0$$

Proof

From (6.5)

$$R_{\alpha\alpha t} = \frac{2t^2\bar{T}(3-4t+t^2\alpha)}{(1-\alpha t)^4} \tag{6.8}$$

which implies

$$R_{\alpha\alpha t} \gtreqless 0 \iff 3-4t+t^2\alpha \gtreqless 0 \tag{6.9}$$

Substituting (6.7) into (6.9) the sign of $R_{\alpha\alpha t}$ along the locus of points satisfying $R_t = 0$ is given by

$$\operatorname{sign}\left\{R_{\alpha\alpha t}\Big|_{R_t=0}\right\} = \operatorname{sign}\left\{3-4\left[\frac{1-\sqrt{(1-\alpha)}}{\alpha}\right] + \alpha\left[\frac{1-\sqrt{(1-\alpha)}}{\alpha}\right]^2\right\}$$

$$= \operatorname{sign}\left\{\frac{2}{\alpha}\left[\sqrt{(1-\alpha)} - (1-\alpha)\right]\right\} \tag{6.10}$$

Therefore

$$0 < \alpha < 1 \implies R_{\alpha\alpha t}\Big|_{R_t = 0} > 0 \tag{6.11}$$

The proposition is then true by Theorem 1.

□

This shows that for all feasible values of estimates of α, under uncertainty the optimal tax formula (6.7) will produce tax rates which are too low. Thus, in fact Broome's theorem should say that the optimal tax rate is no less than 58.6%. Indeed, it may be closer to the rate of 66.6% which he proposes is valid for a more general utility function, and may be easier for politicians to advocate.

Proposition 1.2

Under Cobb Douglas labour supply, the solution to the certainty equivalent problem entails upward adjustment of α above the mean of the distribution

$$0 < \alpha < 1 \quad \Rightarrow \quad \alpha^* - \hat{\alpha} > 0$$

Proof

Differentiating (6.5)

$$R_{t\alpha} = \frac{t\bar{T}(2-3t+\alpha t^2)}{(1-\alpha t)^3} \tag{6.12}$$

which implies

$$R_{t\alpha} \gtreqless 0 \iff 2-3t+\alpha t^2 \gtreqless 0 \tag{6.13}$$

Substituting (6.7) into (6.13), the sign of $R_{t\alpha}$ along the locus of points satisfying $R_t = 0$ is given by

$$\text{sign}\left\{R_{t\alpha}\Big|_{R_t = 0}\right\} = \text{sign}\left\{2-3\left[\frac{1-\sqrt{(1-\alpha)}}{\alpha}\right] + \alpha\left[\frac{1-\sqrt{(1-\alpha)}}{\alpha}\right]^2\right\}$$

$$= \text{sign}\left\{\frac{1}{\alpha}\left[\sqrt{(1-\alpha)} - (1-\alpha)\right]\right\} \tag{6.14}$$

Then, using proposition 1.1

$$0 < \alpha < 1 \quad \Rightarrow \quad \left.\frac{R_{t\alpha\alpha}}{R_{t\alpha}}\right|_{R_t = 0} > 0 \qquad , \tag{6.15}$$

Therefore the proposition follow: from Theorem 2.

□

Proposition 1.2 suggests that it may be possible to retain the simple and attractive optimal tax formula given by equation (6.7). Instead of including complicated terms involving variances, it is possible to make the appropriate adjustment in terms of α. Given that the estimate of $\alpha = 0.5$ is uncertain, it may be regarded as a lower bound: some α greater than 0.5 will yield the optimal tax rate under uncertainty.

Estimates of the magnitudes of the required adjustments to t (or to α) may be derived using approximations 1 and 2. Figure 6.1 illustrates for all values of α the optimal tax rate $t(\alpha, \text{var}\alpha = 0)$ if α is certain, and the expected revenue maximising tax rate $t^*(\alpha, \text{var}\alpha = .1)$ given the variance of $\alpha = .1$. The former is calculated directly from (6.7) and the latt-. by substituting the derivatives of (6.5) into approximation 1 to give

$$\left[1 + t^*(\alpha t^* - 2)\right](1 - \alpha t^*)^2 + (2t^{*2}(3 - 4t^* + t^{*2}\alpha)).1 = 0 \tag{6.16}$$

having normalised $\bar{T}$ to unity. The figure of .1 is chosen for the variance for the following reason. Given that α is between 0 and 1, its greatest possible variance is .25. This occurs when α takes the values of 0 and 1 each with probability .5. Therefore, the value of .1 is chosen because it is roughly in the middle of the feasible range.

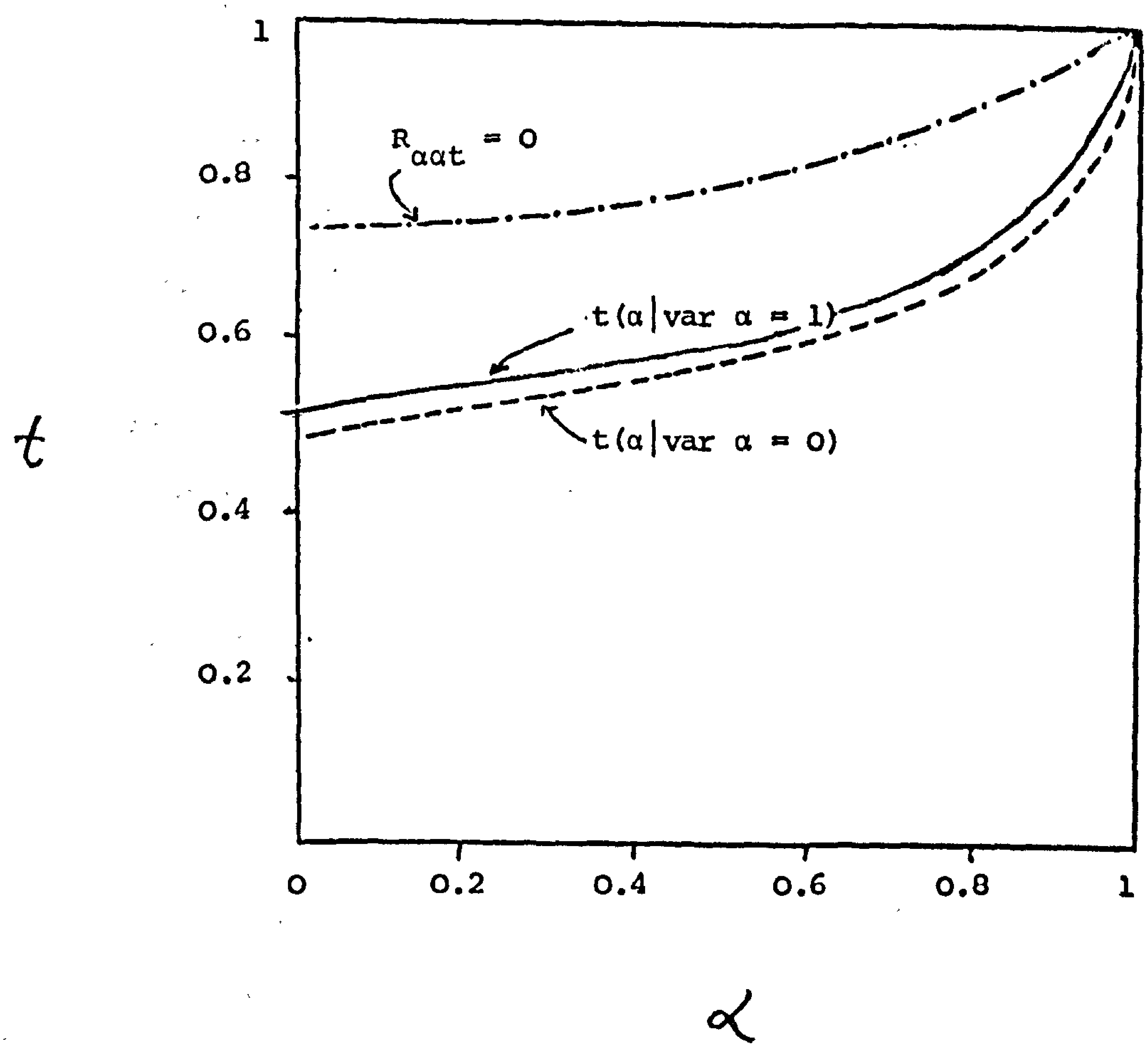


FIGURE 6.1 : Optimal Choice of Income Tax Rate, Cobb Douglas example with $\mathrm{var}\alpha = .1$.

In Figure 6.1, the adjustment of t for uncertainty is the vertical distance between the two lines. For values of α below 0.8 the increase in the tax rate to allow for uncertainty is between 3% and 5%. Then as the expenditure share of leisure approaches zero, both t, and t^* tend to 100%, so the difference tends to zero. Thus, Broome's important theorem would require an income tax rate of 61.2% with a variance of α of .1.

The solution to the certainty equivalent problem, the adjustment of α for uncertainty, may be represented in the figure as the horizontal distance between the two lines. Taking the Broome theorem as an example, if the estimated value of α is 0.5, with a variance of one, the optimal tax rate is .61 . The same tax rate would result from regarding an α value of about .6 as known with certainty. Thus the approximate solution to the certainty equivalent problem in this case would be $\alpha^* - \alpha = .1$ The required percentage increase in α (that is $(\alpha^*-\bar{\alpha})/\bar{\alpha}$) is 20%

6.2.2 Model 2 : Linear Labour Supply with Diversity of Wages

The clear-cut results with the Cobb Douglas utility function do not necessarily apply to other demand systems. The linear labour supply function provides an example of when uncertainty about some parameters matters, and others do not. Also it is possible to use empirical estimates and their standard errors to assess the likely significance of uncertainty adjustments. Assume that labour supply is a linear function of wages w and lump-sum transfers m

$$\ell_h = \alpha w_h(1-t) + \beta m + \gamma \qquad h = 1,\ldots,H \tag{6.17}$$

The associated utility function (see Stern, 1984a) is

$$U_h = \frac{\ell - b}{\beta} \exp\left\{ - \left[1 + \frac{\beta(x+a)}{b-\ell}\right]\right\} \tag{6.18}$$

where

$$b = \frac{\alpha}{\beta} \quad ; \quad a = \frac{\gamma}{\beta} - \frac{\alpha}{\beta^2}$$

The government wishes to maximise revenue, therefore the objective function is simply

$$m = \frac{R}{H} = \sum_h t w_h \ell_h / H \tag{6.19}$$

Substituting (6.17) into (6.19) it is possible to express this in terms of instruments and parameters

$$m = \frac{1}{H} \sum_h t w_h \left[\alpha w_h (1-t) + \beta m + \gamma\right]$$

$$= \frac{t\left[\alpha s(1-t) + \gamma\right]}{v - t\beta} \equiv \frac{A}{v - t\beta} \tag{6.20}$$

where $v = \frac{H}{\Sigma w}$; $s = \frac{\Sigma w^2}{\Sigma w}$, and A is defined above.

Clearly the per capita revenue function (6.20) is linear in α and γ, therefore by Theorem 1, uncertainty from either of these sources does not affect the optimal choice of t. Hence,

Proposition 2.1

Under linear labour supply, uncertainty about either the coefficients on wages or the intercept does not affect the choice of expected welfare maximising t

$$\frac{\partial t^*}{\partial \mathrm{var}\alpha} = \frac{\partial t^*}{\partial \mathrm{var}\gamma} = 0$$

However, β, the coefficient on non-labour income in the labour supply function appears in the denominator of (6.20), therefore uncertainty about this parameter does affect policy choice; the direction of influence depending on the other labour supply parameters.

Proposition 2.2

Under linear labour supply the introduction of uncertainty about β increases or decreases the optimal level of income taxation depending on the signs and magnitudes of the intercept, and wage coefficient

$$\alpha s(1-t)+\gamma \gtreqless 0 \iff \frac{\partial t^*}{\partial \mathrm{var}\beta} \gtreqless 0$$

Proof:

Under certainty the optimal choice of t is implicitly defined by the first order condition, differentiating (6.20) and simplifying

$$m_t = \frac{t^2\beta\alpha s - 2t\alpha sv + v\alpha s + v\gamma}{(v-t\beta)^2} = 0 \qquad (6.21)$$

As long as the numerator is bounded this is equivalent to

$$t^2\beta\alpha s - 2t\alpha sv + v\alpha s + v\gamma = 0 \qquad (6.22)$$

Also from (6.20)

$$m_{\beta\beta t} = \frac{6Atv - 2t^3\alpha sv + 2t^4\alpha s\beta}{(v-t\beta)^4} \qquad (6.23)$$

hence, substituting for A, collecting terms, and making some cancellations

$$\text{sign}\left\{m_{\beta\beta t}\right\} = \text{sign}\left\{3\alpha sv - 4\alpha stv + 3\gamma v + \alpha st^2\beta\right\} \qquad (6.24)$$

Subtracting (6.22) from (6.24) the sign of $m_{\beta\beta t}$ along the locus of points satisfying $m_t = 0$ is given by

$$m_{\beta\beta t}\Big|_{m_t=0} \gtreqless 0 \iff \alpha s(1-t) + \gamma \gtreqless 0 \qquad (6.25)$$

□

This means that if labour supply is upward sloping with a positive intercept $(\alpha,\gamma > 0)$ the optimal tax rate increases with uncertainty, but if the intercept is negative the outcome depends on the magnitudes of the parameters. A backward-sloping labour supply curve could not have a negative intercept (this would allow only negative supplies) therefore the outcome is again indeterminate, taxes increasing with uncertainty only if the intercept is very large relative to the slope.

The solution to the certainty equivalent problem is more complicated because it depends on the second as well as the third derivatives of the revenue function. However, it is possible to characterise the solution given parameters of certain signs. This is summarised by:

Proposition 2.3

Under linear labour supply with positive intercept, and non-positive coefficient on transfers, the solution to the certainty equivalent problem entails an upward adjustment of β above the mean of the distribution.

$$(\alpha, \gamma > 0, \quad \beta < 0) \quad \Rightarrow \beta^{*} - \bar{\beta} > 0$$

Proof

From (6.20)

$$m_{\beta t} = \frac{2Av + \beta t^{3} s\alpha + t^{2} s\alpha v}{(v - \beta t)^{3}} \tag{6.26}$$

If $\beta \leq 0$, substituting from (6.22) yields

$$m_{\beta t}\Big|_{m_t = 0} \gtreqless 0 \iff \alpha s(1-t) + \gamma \gtreqless 0 \qquad (6.27)$$

Assuming that $\alpha, \gamma > 0$, and using (6.25)

$$\frac{m_{\beta\beta t}}{m_{\beta t}} > 0 \qquad (6.28)$$

Therefore Theorem 2 provides the required proposition.

□

It would be very surprising if the coefficient on non-labour income were positive in empirical studies - such results would probably not be published - however a negative slope $(\gamma < 0)$ cannot be ruled out a priori. Thus proposition 2.3 applies only for upward sloping linear supply functions.

Consider the case when the government is required to use taxation to raise a given amount of revenue rather than to maximise tax receipts. The introduction of uncertainty rules out the exact fulfillment of the revenue target ex post, however it may still be imposed as an ex ante planning requirement. If the revenue function is concave in the uncertain parameter, an increase in uncertainty would tend to reduce expected revenue and require an increase in taxation (given that taxes are not above the level required to maximise revenue) to meet the planning target. With the linear labour supply function, the curvature of the revenue function depends on the signs and sizes of the parameters.

Proposition 2.4

With linear labour supply the lowest tax rate which raises $\bar{R}$ (to a second order approximation) under uncertainty is less than that which collects $\bar{R}$ under certainty if

$$\alpha \bar{s}(1-t) + \gamma > 0$$

Proof

From (6.20)

$$m_{\beta\beta} = \frac{2At^2}{(v-t\beta)^3} \tag{6.29}$$

Since β is non-positive, this has the sign of A, and the proposition follows from approximation 3.

□

6.2.3 Model 3 : Simulation using Quadratic Labour Supply Function

The previous two sections showed how adjustments for uncertainty may be made, and examined their theoretical properties. Some numerical results were presented for the Broome theorem but their relationship to my empirical data was rather tenuous. In order to assess the likely magnitudes involved, the labour supply function estimated by Brown et al (1976) is taken as the basis for this example. Their preferred specification is quadratic:

$$\ell = a + b(1-t)w + c(1-t)^2w^2 + dI + eI^2 + fIw(1-t) + gm \qquad (6.30)$$

It is convenient for this exercise because it is in levels, and is linear in non-labour income. I stands for intercept, and is included by Brown et al to capture non-linearities in the budget constraint which arise from the overtime premium ϕ and the tax threshold S. I is defined as

$$I = tT + w(1-\phi)(1-t)S \qquad (6.31)$$

where T is the average working week. Substituting into the labour supply function summing over households and rearranging for m yields

$$m = \frac{t\,\Pi}{\sigma_1 - gt} \qquad (6.32)$$

where $\Pi = a + dtT + et^2T^2 + \left\{(1-t)\left[(d+2etT)(1-\phi)S+b+ftT\right]\right\}\sigma_2$

$$+ \left\{(1-t)^2\left[e(1-\phi)^2S^2 + c + f(1-\phi)(1-t)S\right]\right\}\sigma_3$$

$$\sigma_1 = \frac{H}{\Sigma w}$$

$$\sigma_2 = \frac{\Sigma w^2}{\Sigma w}$$

$$\sigma_3 = \frac{\Sigma w^3}{\Sigma w}$$

Clearly, the only parameter which does not enter the per capita revenue function linearly is g, the cofficient on non-labour income. The expected revenue function, given uncertainty about g may be approximated by the Taylor expansion, around the expected g

$$\overline{Em} \simeq m_{-} + \frac{1}{2} m_{gg} \text{var}(g) \qquad (6.33)$$

These two functions are plotted against t, using the parameter estimates from Brown et al, see Figure A.2. Incorporating the variance of their estimate of g increases the revenue maximising tax rate by 6.3%.

Looking at the upward sloping sections of the lines in Figure 6.2, it is apparent that uncertainty makes little difference to the tax rates which raise small amounts of revenue (up to £15 per capita per week). However, if the revenue requirement was in the order of £23 per capita per week, adjustment for uncertainty would lead to a reduction in the tax rate of about 10%.

The sources of the parameter values used in the construction of Figure 6.2 are as follows:

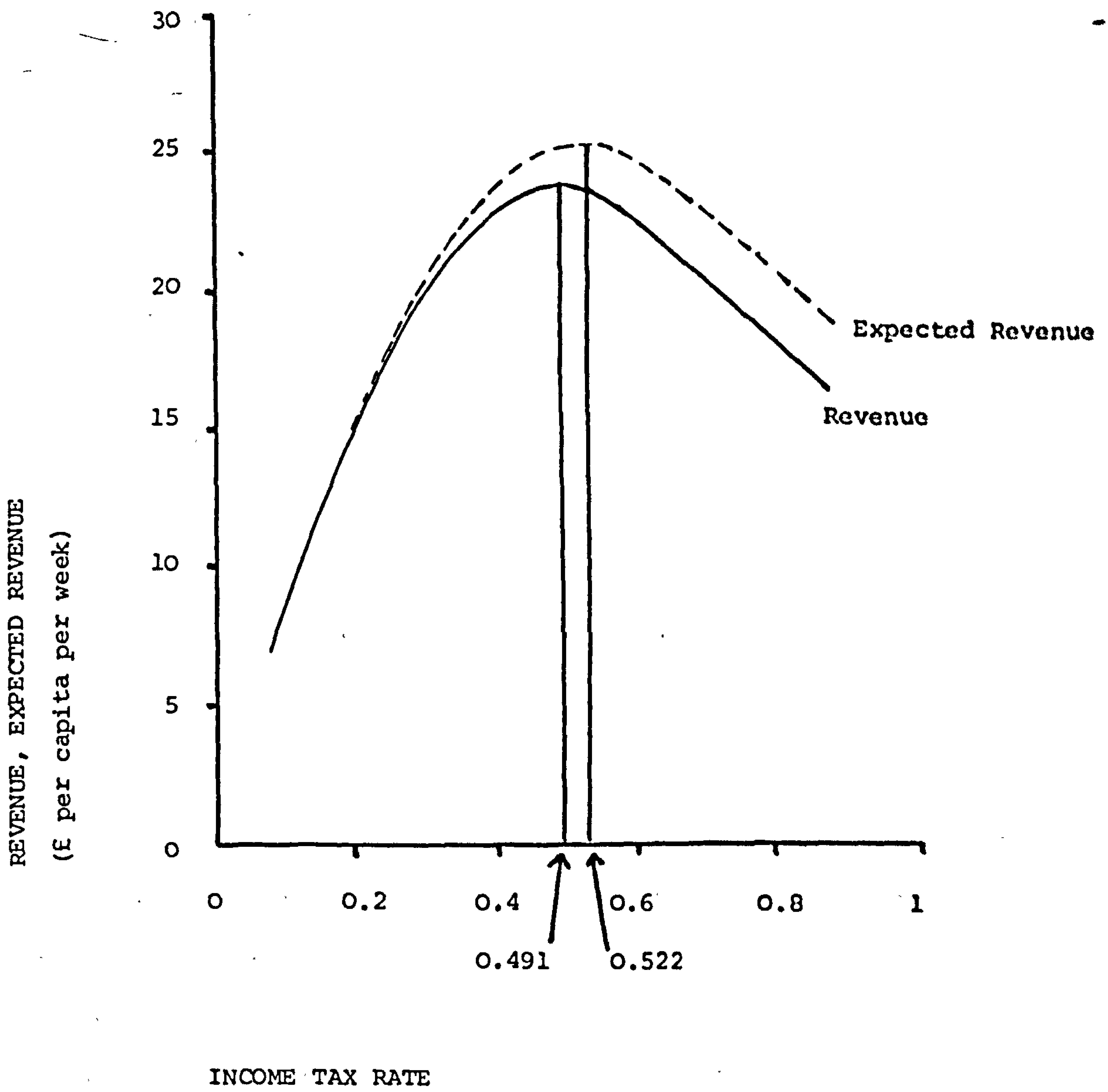


FIGURE 6.2 : Tax Simulations Based on Quadratic Labour Supply Function, Brown, Levin and Ulph (1976)

Taken directly from Brown et al, Table 1, column 1, p.274.

$a = 55.3$
$b = -14.3$
$c = 3.63$
$d = -0.959$
$e = 0.0105$
$f = \bar{0}.644$
$g = -0.114$
$T = 47.4$

Inferred from reported elasticities

$\phi = 1.5$
$s = 128$
$\sigma_1 = 2$

Assumed

$\sigma_2 = 1.25$
$\sigma_3 = 3.9$

The values for σ_2 and σ_3 are consistent with a log-normal distribution for w with mean inferred from reported elasticities, and assumed variance 0.375. (See Newbery and Stiglitz, 1981, p.89, for details of the relationship between the moments of the log-normal distribution, and the mean and variance of w.) The reported standard error for g is 0.048, and the sample size is 434, therefore the variance of g is 1.0.

6.2.4 Model 4 : Ramsey Taxes

Let consumer behaviour be represented by the linear expenditure system with unitary elasticity of substitution between goods and leisure. The direct utility function is

$$U = \sum_{i=0}^{n} \beta_i \log(q_i - \gamma_i) \tag{6.34}$$

where β_i are marginal propensities and γ_i are necessary or subsistence expenditures. Endogenous labour supply may be incorporated in this model by defining $-\gamma_0$ as the time endowment, and $-q_0$ as the supply of labour. Utility maximisation, given no lump sum income yields demand functions

$$q_i = \gamma_i + \frac{\beta i}{p_i} (- p.\gamma) \tag{6.35}$$

The vector product $(- p.\gamma)$ is the income which would result from working all endowed hours minus "necessary" expenditure on goods, and is assumed to be positive. p_0 is the fixed wage rate and p_i is the consumer price of good i. Producer prices are held constant at unity, therefore $p_i = 1 + t_i$. Substituting the demands into the utility function yields the indirect utility function

$$V = \sum_{i=0}^{n} \beta_i \log \left[\frac{\beta_i}{p_i} (- p.\gamma) \right] \tag{6.36}$$

Chapter Five drew a distinction between ex ante and ex post revenue constraints which the government might face. The former is easier to analyse because the revenue constraint contains only fixed quantities, such as expected demands. However, when one instrument is determined ex post, to balance the books, the optimal tax problem must take into account the curvature of the constraint with respect to the unknown parameter. The results for the ex ante constraint are presented first. One quite general result emerges from this case, namely that even in the n commodity model, uncertainty about any one of the β's does not matter.

Proposition 4.1

With ex ante revenue constraint, optimal taxes are invariant with uncertainty about any one of the β's.

Proof

Differentiating (6.36)

$$V_{\beta j} = 1 + \log \left[\frac{\beta_j}{p_j} (-p.\gamma)\right] \tag{6.37}$$

and

$$V_{\beta j \beta j} = \frac{1}{\beta_j} \tag{6.38}$$

Therefore

$$V_{\beta j \beta j t i} = 0 \text{ for all } j, i \tag{6.39}$$

□

However, it is not possible to make such generalisations about uncertainty surrounding the γ's. Differentiating (6.36) gives

$$V_{\gamma j \gamma j} = - \frac{p_j^2}{(p.\gamma)^2} \tag{6.40}$$

which clearly does depend on the taxes. To define precisely what effect uncertainty about any of the γ's has on the optimal tax on each good, it is necessary to restrict attention to the special case where only two goods are taxable. This is the model developed by Corlett and Hague (1953). As indicated in Chapter Five this restriction is required so that, taking account of the revenue constraint, only one independent instrument is available to the government, and thus the Diamond and Stiglitz theorem may be applied. Otherwise, the outcome would depend on a matrix of second derivatives, the inversion of which would render the problem analytically intractable.

<u>Proposition 4.2</u>

<u>With an ex ante revenue constraint the optimal tax on good 1 decreases with uncertainty about</u> γ_1 <u>but increases with uncertainty about</u> γ_2.

$$\frac{\partial t_1^*}{\partial \text{var}\gamma_1} < 0 \; ; \quad \frac{\partial t_1^*}{\partial \text{var}\gamma_2} > 0$$

Proof

Differentiating (6.40) with respect to t_1

$$\frac{d}{dt_1} V_{\gamma j \gamma j} = \frac{\partial}{\partial t_1} V_{\gamma j \gamma j} + \frac{\partial}{\partial t_2} V_{\gamma j \gamma j} \frac{dt_2}{dt_1} \tag{6.41}$$

From the budget constraint, eq. (5.20) Chapter Five

$$\frac{dt_2}{dt_1} = - \frac{q_1 + \sum_{j=1}^{2} t_j \frac{dq_j}{dt_1}}{q_2 + \sum_{j=1}^{2} t_j \frac{dq_j}{dt_2}} \tag{6.42}$$

Using Cournot aggregation (Deaton and Muellbauer, 1980, p.16)

$$\frac{dt_2}{dt_1} = - \frac{\sum_{j=0}^{2} \frac{dq_j}{dt_1}}{\sum_{j=0}^{2} \frac{dq_j}{dt_2}} \tag{6.43}$$

Note also that this depends on the previous assumption that labour is not taxed, and producer prices are fixed and normalised to unity. Using (6.40) and (6.43) in (6.41)

$$\frac{d}{dt_1} V_{\gamma 1 \gamma 1} = \frac{2p_1}{(p.\gamma)^3} \left[(p_1\gamma_1 - p.\gamma) \Sigma \frac{dq_j}{dt_2} - p_1\gamma_2 \Sigma \frac{dq_j}{dt_1} \right] / \Sigma \frac{dq_j}{dt_2} \tag{6.44}$$

From (6.35)

$$\sum_{j=0}^{2} \frac{dq_j}{dt_i} = - \gamma_i \sum_{j=0}^{2} \frac{\beta_j}{p_j} + \frac{\beta_i}{p_i^2} (p.\gamma) \qquad (6.45)$$

Therefore, using (6.45) in (6.44), and simplifying

$$\frac{d}{dt_1} V_{\gamma 1 \gamma 1} = \frac{2p_1}{(p.\gamma)^3} \left\{ p.\gamma \left[\gamma_2 \left(\frac{\beta_0}{p_0} + \frac{\beta_2}{p_2} \right) + \frac{p_1 \gamma_1 \beta_2}{p_2^2} \right] - (p.\gamma)^2 \frac{\beta_2}{p_2^2} \right\} / \sum \frac{dq_j}{dt_2} < 0 \qquad (6.46)$$

Thus the optimal tax on good one decreases with uncertainty about γ_1

$$\frac{\partial t_I^*}{\partial var\gamma_1} < 0 \qquad (6.47)$$

A symmetric argument applies to t_2^* and γ_2

$$\frac{\partial t_2^*}{\partial var\gamma_2} < 0 \qquad (6.48)$$

Since the linear expenditure system rules out inferior goods (Deaton and Muellbauer, 1980, p.139) the revenue constraint requires that t_1 is increased if t_2 has been reduced

$$\frac{dt_1}{dt_2} < 0 \qquad (6.49)$$

Inspection of (6.43) and (6.45) confirms that this is true.

Therefore

$$\left(\frac{dt_2^*}{\partial \mathrm{var}\gamma_2} < 0 \right) \Rightarrow \left(\frac{\partial t_1^*}{\partial \mathrm{var}\gamma_2} > 0 \right) \tag{6.50}$$

□

The interpretation of this result relies on the concavity of the welfare function. When the revenue constraint applies ex ante it is unlikely to be fulfilled precisely when demands are revealed. Then concavity implies that a revenue surplus will entail a greater loss of welfare than the gain which would be associated with a deficit. To hedge against this risk, the deployment of tax instruments is shifted towards goods whose demands are more certain. Thus if γ_1 is uncertain the demand for q_1 is relatively risky so taxation, relative to a situation of perfect knowledge is shifted away from good one, increasing the burden on good two. However no such result applies to γ_0.

With an ex ante revenue constraint, the effect of uncertainty about γ_0 depends on the parameter values. Following the method of the previous proof

$$\frac{d}{dt_1} V_{\gamma o \gamma o} = \frac{2\, p_o^2}{(p.\gamma)^2} \left[\frac{\gamma_1 \beta_2}{p_2^2} - \frac{\gamma_2 \beta_1}{p_1^2} \right] / \Sigma \frac{dq_1}{dt_2} \tag{6.51}$$

Therefore

Proposition 4.3

$$\frac{\partial t_1^*}{\partial \mathrm{var}\gamma_0} \gtreqless 0 \quad \text{if} \quad \gamma_2\beta_1 p_2^2 \gtreqless \gamma_1\beta_2 p_1^2$$

The question of whether to regard an uncertain parameter estimate as an upper or lower bound for a particular policy application is answered by the solution to the certainty equivalent problem. The desired direction of adjustment of the optimal tax rate has already been established. Therefore, the certainty equivalent problem requires in addition knowledge of how the optimal tax rate would be affected by changing the parameter, namely the sign of $dt^*/d\Theta$. Once again, this problem may be reduced to that of signing the appropriate derivatives, and the following proposition emerges.

Proposition 4.4

With an ex ante revenue constraint the certainty equivalent estimate is larger than the standard estimate for γ_1 and γ_2.

$$\gamma_i^* - \bar{\gamma}_i > 0 \quad i = 1,2$$

Proof

Differentiating (6.36) yields

$$V_{\gamma 1} = \frac{p_1}{p.\gamma} \tag{6.52}$$

Differentiating with respect to t_1

$$\frac{d}{dt_1} V_{\gamma 1} = \frac{\partial}{\partial t_1} V_{\gamma 1} + \frac{\partial}{\partial t_2} V_{\gamma 2} \frac{dt_2}{dt_1} \tag{6.53}$$

Evaluating the derivatives, and using (6.43) and (6.45)

$$\frac{d}{dt_1} V_{\gamma 1} = \frac{1}{(p.\gamma)^2} \left\{ \left[\frac{p_1 \gamma_1 \beta_2}{p_2^2} + \gamma_2 \left(\frac{\beta_0}{p_0} + \frac{\beta_2}{p_2} \right) \right] (-\, p.\gamma) + \frac{\beta_2}{p_2^2} (p.\gamma)^2 \right\} / \Sigma \, \frac{dq_i}{dt_2} < 0 \tag{6.54}$$

Proposition 4.2 showed that

$$\frac{d}{dt_1} V_{\gamma 1 \gamma 1} < 0 \tag{6.55}$$

Therefore

$$\frac{d}{dt_1} V_{\gamma 1 \gamma 1} \Big/ \frac{d}{dt_1} V_{\gamma 1} > 0 \tag{6.56}$$

A symmetric argument applies to γ_2

$$\frac{d}{dt_1} V_{\gamma 2 \gamma 2} \Big/ \frac{d}{dt_1} V_{\gamma 2} > 0 \tag{6.57}$$

Thus the proposition follows from Theorem 2.

□

The preceding results under the ex ante revenue constraint may be contrasted with the analogous outcomes given that the constraint must bind ex post. A slightly different method is used in the derivations, as explained in Chapter Five. This requires imposition of the budget constraint before examining the concavity of the programme. Differentiating (6.36) with respect to t_1 yields

$$\frac{dV}{dt_1} = \frac{\partial V}{\partial t_1} + \frac{\partial V}{\partial t_2}\frac{dt_2}{dt_1} \tag{6.58}$$

Using Roy's identity (Deaton and Muellbauer, 1980, p.41), this may be expressed as

$$\frac{dV}{dt_1} = -\alpha\left[q_1 + q_2\frac{dt_2}{dt_1}\right] \tag{6.59}$$

where α is the derivative of the indirect utility function with respect to lump-sum income. Now, using the derivative of the budget constraint (6.43)

$$\frac{dV}{dt_1} = \frac{H}{K} \tag{6.60}$$

where

$$H = -\alpha\left[q_1\sum_{j=0}^{2}\frac{dq_j}{dt_2} - q_2\sum_{j=0}^{2}\frac{dq_j}{dt_1}\right]$$

and

$$K = \sum_{j=0}^{2}\frac{dq_j}{dt_2}$$

Notice that at the optimum, the contents of the brackets in H is equal to zero. The task is now to examine the convexity of (6.59) in the region of the optimum, thus some simplifications will be gained by setting $H = 0$. Differentiating (6.60) with respect to some parameter θ, and using $H = 0$,

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{H}{K}\right) = \frac{1}{K}\left[\frac{\partial^2 H}{\partial\theta^2} - \frac{2}{K}\cdot\frac{\partial H}{\partial\theta}\ \frac{\partial K}{\partial\theta}\right] \tag{6.61}$$

By mechanically implementing this formula it is possible to derive the effects of uncertainty about any of the parameters. The procedure is as follows: firstly, write H, and K in terms of the parameters by using (6.35) and its derivatives

$$K = -\gamma_2 \Sigma \frac{\beta_j}{p_j} + \frac{\beta_2}{p_2^2}\ (p.\gamma) < 0 \tag{6.62}$$

$$H = \gamma_1 \frac{\beta_2}{p_2^2}\ (1 - p_2\ \Sigma \frac{\beta_j}{p_j}) - \frac{\gamma_2\beta_1}{p_1^2}\ (1 - p_1\ \Sigma \frac{\beta_j}{p_j}) \tag{6.63}$$

$$+ \frac{\beta_1\beta_2}{p_2^2 p_1^2}\ (p_2 - p_1)\ (p.\gamma)$$

Secondly, find the first derivatives of H and K, and the second derivatives of H with respect to each of the parameters. These are displayed in Table 6.1. Finally, substitute these derivatives into (6.61) and hence prove, by Theorem 1 in Chapter Four, the propositions listed in Table 6.2.

TABLE 6.1 : Derivatives of H and K

	∂K	∂H	$\partial^2 H$
$\partial\gamma_0$	$\frac{\beta_2}{p_2^2} p_0 > 0$	$\frac{\beta_1\beta_2}{p_2^2 p_1^2} (p_2 - p_1)\, p_0$	0
$\partial\gamma_1$	$\frac{\beta_2}{p_2^2} p_1 > 0$	$\frac{\beta_2}{p_2^2}\, \beta_0\, (1 - \frac{p_2}{p_0}) < 0$	0
$\partial\gamma_2$	$-\frac{\beta_0}{p_0} - \frac{\beta_1}{p_1} < 0$	$-\frac{\beta_1}{p_1^2}\, \beta_0\, (1 - \frac{p_1}{p_0}) > 0$ 1/	0
$\partial\beta_0$	$-\frac{\gamma_2}{p_0} < 0$	$\frac{1}{p_0} \left[\frac{\gamma_2\beta_1}{p_1} - \frac{\gamma_1\beta_2}{p_2}\right]$	0
$\partial\beta_1$	$-\frac{\gamma_2}{p_1} < 0$	$-\frac{\gamma_1\beta_2}{p_1 p_2} - \frac{\gamma_2}{p_1^2} (1 - p_1\, \Sigma\, \frac{\beta_j}{p_j} - \beta_1)$	$2\, \frac{\gamma_2}{p_1^2} > 0$
$\partial\beta_1^2$		$+ \frac{\beta_2}{p_2^2 p_1^2} (p_2 - p_1)\, (p.\gamma)$	
$\partial\beta_2$	$-\frac{\gamma_2}{p_2} + \frac{p.\gamma}{p_2^2} < 0$	$+ \frac{\gamma_2\beta_1}{p_1 p_2} + \frac{\gamma_1}{p_2^2} (1 - p_2\, \Sigma \frac{\beta_j}{p_j} - \beta_2)$	$-2\, \frac{\gamma_1}{p_2^2} < 0$
$\partial\beta_2^2$		$+ \frac{\beta_1}{p_2^2 p_1^2} (p_2 - p_1)\, (p.\gamma)$	

1/ Note that this requires $\sum_{i=0}^{2} \beta_i = 1.$

TABLE 6.2 : Propositions 4.5 - 4.10

Proposition	With an ex post constraint, the optimal tax on good one:
4.5	increases (decreases) with uncertainty about γ_0 if $t_1 > (<)\ t_2$
4.6	decreases with uncertainty about γ_1
4.7	decreases with uncertainty about γ_2
4.8	increases (decreases) with uncertainty about β_0 if $p_2\gamma_2\beta_1 > (<)\ p_1\gamma_1\beta_2$
4.9	decreases with uncertainty about β_1
4.10	increases with uncertainty about β_2 if $t_1 > t_2$

The striking feature of these results, in contrast to the analysis of the ex ante budget constraint, is that uncertainty about any of the parameters matters. No longer is it possible to say that variance of the β's does not affect optimal taxation. This is hardly surprising: the revenue constraint now includes uncertain quantities whereas previously it was expressed in terms of their expectations. Therefore, the ex post revenue constraint adds another element of uncertainty to the problem. Comparison of the ex ante and ex post results requires some careful interpretation since the meaning of t_2 is rather different in the two cases. With an ex ante constraint, t_2 is simply the tax rate which is expected to balance the budget given the choice of t_1. It is an expression containing the expectation of the uncertain parameter which is in turn a fixed quantity. However, in the ex post case t_2 is the tax rate which actually balances the budget when the true value of the uncertain parameter is revealed, i.e. it is a stochastic variable. Therefore, when proposition 4.5 says that t_1 increases with the variance of γ_0 if t_1 is greater than t_2, this condition depends on the value that the uncertain parameter actually takes. If attention is restricted to cases where variance is small, there will be a wide variety of situations where $t_1 > t_2$, or vice versa, could be asserted with great confidence, and thus provide a guide to the best policy response to changes in uncertainty. If the optimal tax structure is close to uniformity under certainty, then the introduction of a little uncertainty about γ_0 will cause a shift away from uniformity. It is difficult to compare this result with proposition 4.3 where the revenue constraint applies ex ante because firstly, the interpretation of t_2 is different, and secondly, in both sets of conditions t_1 and t_2 are themselves functions of all the parameters of the model, and the possibility of writing them out explicitly was ruled out at an early stage in the analysis.

However, propositions 4.6 and 4.7 are unconditional as is their ex ante counterpart 4.2. There is agreement between the two models that the tax on good one decreases with uncertainty about necessary consumption of good one, but the symmetry of the ex ante case, which requires that the tax on good one decreases with uncertainty about necessary consumption of good two, does not carry over to the ex post model. One possible explanation for this difference is as follows: in the ex ante case, increases in t_1 are necessarily associated with reductions in t_2. A good with an uncertain γ is a risky source of tax revenue, therefore taxation is adjusted towards the safer good. But in the ex post case, there is good reason to adjust t_1 downwards because there is always the chance that t_2 may turn out to be smaller than expected. Thus it is worth risking some mismatch between the tax rates in order to increase the chance that the most favourable outcome, with low taxes on both goods, may come to pass. Clearly, this is not an entirely satisfactory explanation because it does not say why different results apply to different parameters, and does not offer support for the conditions derived. Such an explanation is only to be found in the algebra.

Propositions 4.8 - 4.10 are quite distinct from their ex ante analogue 4.1, since uncertainty about the marginal propensities does matter in the ex post model. Uncertainty about β_1 reduces the optimal tax on good one regardless of parameter values. This is presumably because the tendency to reduce t_1 in response to any uncertainty, and the riskiness of good one as a source of revenue, tend to work in the same direction. Propositions 4.8 and 4.10 state the conditions applying to uncertainty about β_0 and β_2. (Note that t_1 could be increased or decreased by uncertainty about β_2 if $t_2 > t_1$). Once again there are difficulties in interpretation because t and p are quite complicated functions of the parameters of the model.

6.2.5 Model 5 : Uniform taxes

The conditions under which optimal indirect tax rates are identical for all commodities are well known, see for example Sadka (1977), Deaton (1981), Atkinson and Stiglitz (1980). In this section a simple model which satisfies these conditions is examined in order to show that the introduction of parameter uncertainty precludes the uniformity result. The model extends the LES/Corlett-Hague model by introducing a second consumer, and a third tax instrument - lump-sum transfers. The government's objectives may now be written as a function of each consumer's indirect utility

$$\Psi[V_1(t_1,t_2,m),\ V_2(t_1,t_2,m)] \tag{6.64}$$

Sufficient revenue must be raised to cover these lump-sum payments: they are included on the right-hand side of the revenue constraint

$$t_1q_1 + t_2q_2 = R + 2m \tag{6.65}$$

Given the quasi-homothetic structure of the linear expenditure system, and consumers distinguished only by their time endowments, the necessary conditions for optimality under certainty are satisfied by

$$t_1 = t_2 \tag{6.66}$$

This is a special case of Atkinson and Stiglitz (1980) pp.433-4.

When parameter uncertainty is introduced in this model, the outcome will depend not only on the properties of the consumer's indirect utility function, but also on the weights these receive in the social welfare function. The simplest case to consider is equal weights

$$\Psi = V_1(t_1, t_2, m) + V_2(t_1, t_2, m) \tag{6.67}$$

Under this assumption the following proposition about uncertainty about the β's is a trivial extension of proposition 4.1

Proposition 5.1

Uncertainty about the β's does not perturb the uniformity result when the revenue constraint applies ex ante

It is difficult to interpret uniformity in the context of an ex post revenue constraint; t_2 would be determined by unpredictable factors and therefore it makes little sense to assert that the policy maker equates it with t_1. Concentrating on the ex ante constraint, uncertainty about the γ's does matter, but once again it is impossible to derive analytic results when the model has more than one independent choice variable.

One way to proceed is to assume that one of the instruments is determined by a rule which is invariant with uncertainty. For example, lump-sum redistributions may be calculated as if there were perfect knowledge of the required parameters. Then m would be set to satisfy the first order conditions of the constrained programme

$$L = \bar{\Psi} - \lambda(2m + R - t_1q_1 - t_2q_2) \tag{6.68}$$

where all uncertain quantities had been replaced by their expectations. The solution to this programme, $\bar{m}$ is a fixed quantity which may be substituted into (6.67). This effectively reduces the problem to one identical to Ramsey taxation in model 4 and similar propositions apply.

Proposition 5.2

If the revenue constraint applies ex ante, uncertainty about any of the γ's means that the uniformity result no longer applies

γ_1 uncertain $\Rightarrow$ $t_1 < t_2$

γ_2 uncertain $\Rightarrow$ $t_1 > t_2$

γ_0 uncertain $\Rightarrow$ $t_1 \lesseqgtr$ depending on parameter values

6.2.6 Model 6 : Uniform Taxes with Demogrants

The preceeding five models were all analysed by means of the techniques described in Chapter Five. In this model, however, uncertainty about errors in administering the tax system may be incorporated using more simple arguments. These arguments are set out in terms of the correlation between the errors and other variables in the model, and are quite distinct from the methods applied to models one to five. Deaton (1981, p.1253) provides the basic model where the government maximises welfare subject to individual rationality and budget constraints. Deaton and Stern (1986) extended this model to allow the distribution of lump-sum grants (known as demogrants) on the basis of the demographic characteristics of the recipient. When this ability is combined with the empirical assertion that Engel curves are linear, a strong argument emerges in favour of uniform indirect commodity taxes.

The analysis here provides an additional extension to the model. Uncertainty about the parameters representing household characteristics is incorporated in the form of observation errors so that demogrants must be based on imperfect information about household characteristics. The desirability of uniform commodity taxation then depends on the correlation between observation errors and social marginal utilities.

Following Deaton (1981), the welfare function is defined on household utilities

$$W = W(U^1,\ldots,U^H) \tag{6.69}$$

This is maximised subject to two constraints. The government budget constraint

$$\sum_k \sum_h t_k q_k^h = R + \sum_h g^h \tag{6.70}$$

requires revenue from taxation of commodities (t_k is the tax rate on good k, and q_k^h is h's consumption of k) to equal some arbitrary level R plus the amount paid in lump sum grants g^h to households. The individual rationality constraint

$$m^h + p_O^h T^h + g^h = c^h(u^h, p_O^h, p) \quad \text{for all } h \tag{6.71}$$

ensures that households have maximised utility by equating income (from lump-sum endowments m^h, wage income $p_O^h T^h$, and grants) with the expenditure or cost function c^h. In the usual way the Lagrangian is written

$$\mathcal{L} = W + \lambda\left[\Sigma\Sigma t_k q_k^h - R - \Sigma g^h\right] + \Sigma\lambda^h\left[m^h + p_O^h T^h + g^h - c^h\right] \tag{6.72}$$

However, departure from the standard tax problem results from the definition of the grants. They are assumed to have one element, γ_O which is identical across consumers, and a second which is determined by observed household characteristics.

$$g^h = \gamma_O + \sum_j \gamma_j (z^{hj} + e^{hj}) \tag{6.73}$$

where z^{hj} represents the j^{th} characteristic of household h, and e^{hj} the corresponding observation error. Deaton and Stern (1986) assume that z is known with certainty, and thus $e^{hj} = 0$ for all h,j.

Using (6.73) in the Lagrangian, and differentiating, yields first order conditions with the following implications:

$$\frac{\partial \mathcal{L}}{\partial \gamma o} = o \Rightarrow \Sigma \lambda^h = H\lambda \tag{6.74}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma_j} = o \Rightarrow \sum_h \lambda^h (z^{hj} + e^{hj}) = \lambda \sum_h (z^{hj} + e^{hj}) \tag{6.75}$$

$$\frac{\partial \mathcal{L}}{\partial t_i} = o \Rightarrow \sum_k \bar{s}_{ik} t_k = - (\bar{q}_i - q_i^*) \tag{6.76}$$

where bars denote arithmetic means, $\bar{s}_{ik}$ is the average across households of the ik^{th} element of the Slutsky matrix, and *'s denote weighted sums with weights $\lambda^h/(\lambda H)$, so

$$q_i^* = \sum_h \frac{\lambda^h}{\lambda H} q_i^h \tag{6.77}$$

λ^h may be interpreted as the social marginal utility of income accruing to household h.

Now, it is clear that the characteristics of the optimal tax system are intimately related to the structure of preferences (the nature of the demand system generating $\bar{q}, q^*$, and s in the first order conditions).

The case examined here is quasi-homotheticity, that is linear Engel curves with positive intercepts. Deaton (1979) shows that in this case the left-hand side of (6.76) is proportional to $\frac{t_k}{1+t_k}\ \beta_i$ and thus if the right-hand side is proportional to β_i a uniform tax solution is possible.

The definition of quasi-homotheticity (Deaton and Muellbauer, 1980, p.144) requires an indirect utility functions whose numerator is super-numerary income, and whose denominator depends on the Engle curve slopes. Following Deaton and Stern (1986), allowing intercepts to vary across households

$$U^h = \frac{c^h - a^h(p)}{b(p)} \tag{6.78}$$

The variation in $a^h(p)$ is assumed to depend on household characteristics and idiosyncratic differences in tastes

$$a^h(p) = a(p) + \sum_j \alpha^j(p) z^{hj} + \varepsilon^h(p) \tag{6.79}$$

Demand functions are derived from (6.78) and (6.79) using the method given by Deaton and Muellbauer (1980), p.144: rearrange (6.78) for c^h, differentiate with respect to p_i, substitute for U^h from (6.78), and use the derivatives of (6.79) to get

$$q_i^h = \beta_i c^h + (a_i - \beta_i a) + \sum_j (\alpha_i^j - \beta_i \alpha^j) z^{hj} + (\varepsilon_i^h - \beta_i \varepsilon^h) \tag{6.80}$$

where $\beta_i = \frac{b_i}{b}$ and a_i, b_i, α_i^j, and ε_i are partial derivatives with respect to p_i. Hence

$$q_i - q_i^* = \beta_i(\bar{c}-c^*) + \sum_j(\alpha_i^j-\beta\alpha^j)(\bar{z}^j-z^{*j}) + [(\bar{\varepsilon}_i-\varepsilon_i^*) - \beta_i\ (\bar{\varepsilon}-\varepsilon^*)] \quad (6.81)$$

From this expression it is clear that $\bar{q}_i - q_i^*$ will be proportional to β_i if the last two terms are zero. The following propositions use this to derive conditions under which a uniform tax solution is possible. The first, which was proposed by Deaton and Stern (1986) assumes no observation errors.

Proposition 6.1 (Deaton and Stern, 1986)

If there are no observation errors ($e^{hj} = 0$ for all h,j) the uniform tax solution holds if idiosyncratic taste differences are not correlated with social marginal utilities of income.

$$\left.\begin{array}{l} e^{hj} = 0 \quad \forall h,j \\ \mathrm{Cov}(\lambda^h,\varepsilon^h) = 0 \end{array}\right\} \Rightarrow \bar{q}_i - q_i^* \propto \beta_i$$

Proof

i) $\quad e = 0 \Rightarrow \sum_h \lambda^h z^{hj} = \lambda \sum_h z^{hj}$ from (6.75)

$$\Rightarrow \bar{z}^j - z^{*j} = 0 \quad (6.82)$$

ii) $\quad Cov(\lambda^h, \varepsilon^h) = 0 \iff \sum_h (\lambda^h - \bar{\lambda})(\varepsilon^h - \bar{\varepsilon}) = 0$

$$\iff \frac{1}{H\lambda} \Sigma(\lambda^h \varepsilon^h + \bar{\lambda}\bar{\varepsilon} - \lambda^h \bar{\varepsilon} - \bar{\lambda}\varepsilon^h) = 0$$

$$\iff \varepsilon^* - \bar{\varepsilon} = 0 \Rightarrow \varepsilon_i^* - \bar{\varepsilon}_i = 0 \tag{6.83}$$

Therefore i) and ii) imply, from (6.81)

$$q_i - q_i^* = \beta_i(\bar{c} - c^*) \propto \beta_i \tag{6.84}$$

$\square$

Proposition 6.2

If there is no correlation between idiosyncratic tastes and social marginal utilities, the proportional tax solution holds if and only if there is no correlation between observation errors and social marginal utilities.

$$\left.\begin{matrix} Cov(\lambda^h, \varepsilon^h) = 0 \\ Cov(\lambda^h, e^{hj}) = 0 \end{matrix}\right\} \iff \bar{q}_i - q_i^* \propto \beta_i$$

Proof

i) $\quad Cov(\lambda^h, \varepsilon^h) = 0 \iff \varepsilon^* - \bar{\varepsilon} = \varepsilon_i^* - \bar{\varepsilon}_i = 0$ (Proposition 6.1)

ii) $\mathrm{Cov}(\lambda^h, e^{hj}) = 0 \iff \sum_h (\lambda^h - \bar{\lambda})(e^{hj} - \bar{e}^j) = 0$

$$\iff e^{*j} - \bar{e}^j = 0$$

$$\iff \bar{z}^j - z^{*j} = 0 \quad \text{using (6.75)}$$

Therefore i) and ii) are equivalent to

$$\bar{q}_i - q_i^* = \beta_i(\bar{c} - c^*) \propto \beta_i$$

□

An example may help to clarify the consequences of these propositions. Consider the demographic characteristic 'number of dependents'. If no grants are available and society wishes to redistribute to those with low incomes and many dependents the only option is to tax more heavily the least important goods in their consumption bundle. However, if demogrants are available there is no need to deviate from proportional commodity taxation unless consumption reveals something about the number of dependents which could not be included in the system of transfers.

With demogrant errors, it is impossible to link transfers precisely to household characteristics. Hence if there is some correlation between the administrative errors and society's preferences about redistribution, it is desirable to tax more heavily those goods which are least important in the large-family consumption bundle.

6.3 DISCUSSION OF RESULTS

The results of the six models examined in this paper are summarised in Table 6.3 Clearly, these examples are all quite special cases of the general problem of taxation under uncertainty, but despite the restrictiveness of their assumptions, they may be of some interest per se. Each example uses a model, and a functional form for preferences which is well known in the literature, thus Table 6.2 may be regarded as an attempt to throw light on previously neglected aspects of well-established models

6.3.1 Linear Income Tax Models

The first three examples are based on the linear income tax model where the government is allowed only one instrument. The tractability of this model is the consequence of quite restrictive assumptions. The first, following Rawls (1971), is that zero weight is allocated to all individuals except the least well off, thus social welfare is identical to the utility of that individual. The second requires that all individuals face the same tax rates and that tax revenue is redistributed equally across the whole population. This assumption ensures that no policy change can change the ranking of individuals; hence the welfare function equals the same individual's utility function before and after the change in income taxation.

TABLE 6.3 : Summary of Results

Model	Instrument	Demand Specification	Result under Certainty	Result under Uncertainty	Certainty equivalent parameter estimates
1	linear income tax	$U = q^{\alpha}(T-\ell)^{(1-\alpha)}$	$t = \frac{1-\sqrt{(1-\alpha)}}{\alpha}$	$\frac{\partial t^*}{\partial var\alpha} > 0$	$\alpha^* > \bar{\alpha}$
2	linear income tax	$\ell = \alpha w(1-t) + \beta m + \gamma$	N/A	$\frac{\partial t^*}{\partial var\alpha} = \frac{\partial t^*}{\partial var\gamma} = 0$ $\text{sign}\left\{\frac{\partial t^*}{\partial var\beta}\right\} = \text{sign}\left\{\alpha w(1-t) + \gamma\right\}$	$\beta^* > \bar{\beta}$ if [illegible]
3	linear income tax	$\ell = a+b(1-t)w+c(1-t)^2w^2 + dI+eI^2+fIw(1-t)+gm$	$t = 0.491$	$t = 0.522$	-
4	indirect taxes t_1, t_2	$U = \sum_{i=0}^{2} \beta_i \log(q_i-\gamma_i)$	N/A	ex ante rev constraint $\frac{\partial t_i^*}{\partial var\beta_j} = 0$ for all i,j $\frac{\partial t_1^*}{\partial var\gamma_1} < 0$; $\frac{\partial t_1^*}{\partial var\gamma_2} > 0$ ex post rev constraint $\frac{\partial t_1^*}{\partial var\gamma_1}, \frac{\partial t_1^*}{\partial var\gamma_2}, \frac{\partial t_1^*}{\partial var\beta_1} < 0$	ex ante rev constraint $\gamma_i^* > \bar{\gamma}_i$ $i = 1,2$
5	indirect taxes t_1, t_2, transfers m	$U = \sum_{i=0}^{2} \beta_i \log(q_i-\gamma_i)$	Uniformity	$\frac{\partial t_i^*}{\partial var\beta_j} = 0$ for all i,j $(var\gamma_j \neq 0) \Rightarrow (t_1^* \neq t_2^*)$ for all j	-
6	indirect taxes t, Demogrants g^h	$U = \frac{q-a(p)}{b(p)}$	Uniformity if $cov(\gamma^h \epsilon^h) = 0$	Uniformity if $cov(\gamma^h,\epsilon^h) = cov(\gamma^h,e^h) = 0$	-

The Rawlsian criterion requires an explanation of why some individuals are better-off than others. Stern (1976) provides a variety of models of individual differences, two of which are used here. In models 2 and 3, differences in skills are represented by a distribution of wages; individuals have identical preferences but the less skillful face lower wages. An alternative approach is used in model 1 which also assumes identical preferences. In model 1, however, wages are constant and individuals differ according to their time endowments.

The combination of linear income taxation with equal redistribution and a Rawlsian welfare function greatly simplifies the government objectives. The utility of the least well-off increases with transfers, therefore the government maximises tax revenue. The tractability of this model has made it the subject of considerable literature, Mirrlees (1971) Atkinson (1976) and Phelps (197[illegible]) have examined it as a particular example of more general models, and have used it for optimal tax calculations. Sheshinski (1972) restricts attention to linear income tax models and describes propositions which apply to more general welfare functions. Feldstein (1973) concentrates on the disincentive effects of income taxation on labour supply. Broome (1975) uses the most restrictive assumption of Cobb Douglas preferences which form the basis of model 1.

Under uncertainty, the results may be summarised as follows: firstly (model 1) with Cobb Douglas preferences, uncertainty about the share parameter increases the optimal income tax rate, and the magnitude of the adjustment is likely to be in the order of 3% to 5%).

The solution to the certainty equivalent problem indicates that the standard parameter estimate may be regarded as a lower bound. In this case the parameter estimate may be increased by about 20% and regarded as known with certainty. Secondly, if the labour supply function is linear in non-labour income (models 2 and 3) only uncertainty about the coefficient on non-labour income matters, the magnitude and direction of uncertainty adjustment depending on parameter values. Model 3 uses the econometric estimates of Brown et al, and finds that the incorporation of uncertainty increases the revenue maximising tax rate by 6.3%.

6.3.2 Indirect Tax Models

Three indirect tax models are examined. The first, model 4, concentrates on efficiency issues, and assumes that there is only one consumer. Preferences are represented by the linear expenditure system. Even in this simple framework it is impossible to derive analytic expressions for the optimal tax rates in terms of the demand parameters. However, the effect of parameter uncertainty may be precisely characterised. When the revenue constraint applies ex ante, at the planning stage, uncertainty about any of the marginal propensities does not affect the optimal tax rates, but uncertainty about the subsistence levels does. In the simplified model with two goods and labour (Corlett and Hague, 1953) the optimal tax on good one increases (decreases) with the variance of the estimate of $\gamma_2 (\gamma_1)$. The effect of uncertainty about γ_1 depends on the magnitudes of the parameters. These results are, perhaps not surprisingly, quite different when the revenue constraint applies ex post. In this case, the effect of uncertainty about β_0, β_2 and γ_0

on t_1^* depends on parameter values, and uncertainty about any of the other parameters reduces t_1^*. Models 5 and 6 both examine uniform commodity taxes, where the model has many consumers and thus incorporates considerations of equity. Sadka (1977) shows that uniformity is optimal under the assumption that either all goods must have the same compensated wage elasticities, or that uniform indirect taxes must maximise the supply of labour. The linear expenditure system has the property of quasi-homotheticity, and given the possibility of identical lump sum grants to all consumers, this is known to imply the optimality of uniform taxes (see Deaton, 1981, Atkinson and Stiglitz, 1980, p.433). Model 5 uses these assumptions and shows that in this context uniformity is not affected by the introduction of uncertainty about any one of the marginal propensities. However when one of the subsistence levels is uncertain it is desirable to have different indirect tax rates across goods.

A more convincing argument in favour of uniform commodity taxation has been suggested by Deaton and Stern (1986). A powerful and frequently used tool for redistribution of income is lump-sum grants based on the demographic characteristics of the recipient. When such a system of 'demogrants' is combined with the empirical assertion that Engle curves are linear, uniform commodity taxation emerges. However, model 6 shows that if demographic characteristics are observed with some error, and that this error is related to society's distributional predelictions, then the uniformity result no longer holds, and it becomes desirable to use the system of indirect taxation to compensate for errors in the administration of demogrants. Stern (1982) discusses errors in the administration of a system of lump-sum grants to compare its merits with optimal non-linear income taxation. The argument here indicates the consequence of such errors for the structure of indirect taxation.

APPENDIX 6.I

Abbott and Ashenfelter (1976) provide augmented LES estimates of a demand system including labour supply. Although their work was based on US data, the accuracy of their estimates may provide an indication of the importance of parameter uncertainty for UK taxation. Taxes on income in the UK yielded a total revenue of £34.6 bn in 1984. In aggregate this may be represented as

$$34.6 = R = tw.\ell$$

where t is the tax rate, w is the gross wage rate (assumed fixed) and ℓ is the number of hours worked. The effect of a change in the rate of taxation is

$$\frac{\partial R}{\partial t} = \ell\left(w + t\,\frac{\partial \ell}{\partial w}\,\frac{w}{\ell}\right) \tag{6.A.1}$$

Abbott and Ashenfelter (1976) show that the elasticity may be written in terms of their parameter estimate, and household income

$$\frac{\partial L}{\partial w}\,\frac{w}{\ell} = -\,\frac{w\ell}{y}\,\beta - (1+\beta)$$

where y is non-labour income. From Financial Statistics, wages and salaries in 1984 amounted to £156 bn, out of a total of £280 bn of income before tax. Combining this with Abbott and Ashenfelter's estimate of

β, the estimated elasticity is

$$\frac{\partial \ell}{\partial w}\,\frac{w}{\ell} = -\frac{156}{124}\ .121 - (1 + .121) = -1.27$$

The tax rate may be approximated by dividing the total tax revenue by wages and salaries. The Department of Employment Gazette gives average hourly earnings for all employees, as £3.65, which implies that £42.7 bn hours were worked in 1984. Using this information in (6.A.1) reveals that $\partial R/\partial t = 143.9$; an increase of 1p in the pound on the income tax rate would yield an extra £1.44 bn to the exchequer, which is a 4.2% increase in tax revenue.

A confidence interval around this revenue effect may be obtained by repeating the calculation using upper and lower bounds for the elasticity. Elementary statistical theory provides the result that

$$\Pr\left(\bar{\beta} - 2\frac{s}{\sqrt{\Sigma q_i^2}} < \beta < \bar{\beta} + 2\,\frac{s}{\sqrt{\Sigma q_i^2}}\right) = 0.95$$

where 2 is the critical value of the t distribution with 38 degrees of freedom. Abbott and Ashenfelter report $\bar{\beta} = 0.121$, $s = 0.0368$, and by inference from their Table III $\sqrt{\Sigma q_i^2} = 4.12$. Thus the 95% confidence interval for β is

$$0.103 < \beta < 0.139$$

which translates into elasticities of

$$-1.31 < \frac{\partial \ell}{\partial w} \frac{w}{\ell} < -1.23$$

The revenue effects of a 1p in the pound increase in the tax rate are then £1.43 bn and £1.44 bn on the basis of the lower and upper limits of the elasticity respectively. The difference is £7.5m.

This may be regarded as an upper bound because: firstly, in practice a number of other sources of information would be brought to bear on the question. And secondly, general equilibrium effects would be expected to erode the revenue increases. The initial reduction in labour supply would reduce disposable income, and may reduce revenue from taxation of expenditure. Of course, general equilibrium effects could go either way, but it seems reasonable to assume that they would to some extent offset the revenue generated by increased income taxation, and reduce the associated confidence interval. However, econometric estimates are usually only regarded as local approximations to more general functions. The accuracy of the approximation would fall rapidly as the size of the policy change increases. Thus the narrow confidence interval described in the preceeding analysis may be appropriate for 1p tax changes, but would increase dramatically if the exercise were scaled up to 10p.

POSTCRIPT

OVERVIEW OF THE FINDINGS

One recurring theme throughout the thesis is the diversity of interactions between imperfect information, and optimal decisions. The simple examples in Chapter One showed that there are quite plausible circumstances where increases in risk will lead the decision-maker to choose a riskier environment. Such an apparently perverse result arises, in this case, from the combination of a particular curvature of the objective function with additive uncertainty, and it is difficult to identify more generally the types of models in which this result applies.

However, for the purpose of the optimal choice of taxes, these considerations may not be empirically significant. Chapter Six shows that it is rather difficult to find examples where parameter uncertainty has a very big effect on optimal tax rates. Although parameter uncertainty, which is necessarily associated with the process of econometric estimation, could in principle have any effect on the best choice of taxes, in practice these effects tend to be small. Therefore, for the standard models of optimal taxation, parameter uncertainty is a small ex post refinement to the model rather than an important issue.

Such refinements may be difficult to implement. A number of issues arise relating to the dimensionality of the problem. With more than one uncertain parameter, some ad hoc measure of overall risk would have to be adopted; and when more than one instrument is available, the problem becomes analytically intractable. Also, Lagrangian techniques are not necessarily valid for the incorporation of a revenue constraint because of the endogeneity of the Lagrange multiplier. These difficulties restricted the analysis to fairly simple models, but nevertheless, it is difficult to refute the assertion that the magnitude of uncertainty adjustments would generally tend to be small in these models.

However, one of the main arguments of the thesis is that dynamics relating to both changes in information over time and sequential policy revisions, may reassert the importance of aspects of policy choice relating to information. The decision theory models of Chapters Two and Three provide examples of situations where this may occur. Firstly, when policy reforms are triggered by changes in a variable describing the state of the economy, optimal decisions may have quite unusual properties, and even when risk is small, the comparative statics of the standard model may be reversed. Secondly, in very risky situations, learning becomes an important consideration, and in extreme cases the optimal policy may entail allocation of some instruments exclusively for short-sighted goals, and others to improve information in the future.

In the numerical simulations presented in Chapter Four, active learning decision rules performed significantly better than risk averse or certainty equivalent strategies. In conjunction with the analytic results obtained in the two/three period models, the optimal active learning strategy has the property that it gives up learning before the variances of the parameter estimates have converged to zero.

Under these circumstances, least squares estimates are unlikely to have desirable properties and there is a strong suggestion that other more sophisticated estimators would also lack desirable asymptotic properties. Of course, in this context, inconcistency would not necessarily mean that the estimator was bad, since its merits may be judged by the combined performance of the estimator and the decision rule.

FURTHER RESEARCH

Three potentially useful topics for further research have arisen during the course of work on this thesis. The first is the examination of tax evasion with errors in the administration of the deterrence mechanism. Existing tax evasion models allow the possibility that evaders may get away with it; the innovation suggested here is that the innocent may be falsely convicted. This would allow the possibility of over-reporting, which in existing models only occurs in repeated games. The object of the analysis would be to derive the effect of false convictions on the optimal deterrence mechanism, and to examine the relationship between the parameters of the deterrence mechanism, and the standard tax instruments. This research would be a natural addition to the optimal tax models of Chapter 6, the main extension being that tax evasion raises issues of incentives, and possibilities of policy games, which were not present in the other models. Also, the notion of administrative errors is the same as that used by Deaton and Stern (1986), and Stern (1982), which appeared in Section 6.2.6 in the context of administration of a system of demogrants.

The second research possibility is also related to taxation. The theory of tax reform used by Ahmad and Stern (1984) takes no account of parameter uncertainty. Although the standard errors of the estimates used in their study of India are so small that uncertainty was not an inportant consideration, it would generally be desirable to allow for uncertainty in their methodology. Whereas Ahmad and Stern calculate directions of welfare improving tax reforms which maintain constant revenue, the question proposed here would be to calculate the probability that any tax change will increase welfare holding expected revenue constant. The objective of this research would be to establish a technique for calculating the direction of reform least likely to decrease (or most likely to increase) welfare.

Finally, the third possible area of further research arises from the Monte Carlo simulations in Chapter 4. Firstly, the restrictive assumption of OLS estimation, which is maintained throughout Chapter 4 could be relaxed. OLS in the situation where the right-hand side variables are determined by a rule based on previous estimates, probably makes poor use of available information. A potentially more attractive estimation strategy may be available in the recent control theory literature on non-linear filtering. Similarly, the type of decision rules used in Chapter 4 are quite arbitrary. A more satisfactory approach to the formulation of approximately optimal decision rules would be to base them on an approximation to the dynamic valuation function.

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